Math 5032 - Homework 8

Due 4/05/06

1. Show that a commutative ring R is primitive if and only if R is a field.
2. Suppose F is an algebraically closed field and D is a division ring that is algebraic over F, with F contained in the center of D. Show that $D = F$.
3. Let $a \in M_n(F)$ be a nilpotent matrix, where F is a not necessarily algebraically closed field. Show that the trace of a is zero. (Hint: Jordan normal form.)
4. (Wedderburn) Suppose that A is a finite-dimensional algebra over a field F and that A has an F-basis consisting of nilpotent elements. Show that A is nilpotent. (Show first that F can be assumed algebraically closed by considering $A^K = K \otimes_F A$, K an algebraic closure of F. Use induction on $\dim_F A$. Suppose result is true for all algebras of dimension less than n and let A have dimension n over F. Suppose the result is false for A. Playing with $J(A)$ and the induction hypothesis show that A is semisimple. Then, using the induction hypothesis again show that A is simple, hence a matrix algebra. Now use the basis of nilpotent elements to see that all elements have trace 0 and arrive at a contradiction.)
5. (Jennings) Suppose G is a finite p-group and F is a field of characteristic p. Show that $J = J(FG)$ has $\{x - 1 : 1 \neq x \in G\}$ as an F-basis, so $\dim_F J = |G| - 1$. (Hint: Let W be the F-subspace spanned by all $x - 1$. Show that W is an ideal and each $x - 1$ is nilpotent. Apply the previous exercise.)
6. Suppose F is an algebraically closed field and A is a finite-dimensional semisimple F-algebra. Show that the dimension of the center of A is equal to the number of simple direct summands of A.
7. Let $F = \mathbb{Q}$, let G_1 be a cyclic group of order 4, and let G_2 be the Klein's 4-group. Show that the group algebras $\mathbb{C}G_1$ and $\mathbb{C}G_2$ are isomorphic $\mathbb{C}$-algebras. Recall that Klein's 4-group consists of the four matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

8. Let $F = \mathbb{Z}/(2)$, let G_1 and G_2 be as in the previous problem. Show that FG_1 and FG_2 are not isomorphic. (Hint: Investigate the "degree of nilpotence" of their Jacobson radicals. See exercise 5.)