Math 5032 - Homework 9

Due 4/17/06

1. Given any two representations T and S of a finite group G on finite dimensional complex vector spaces, show that the inner product

$$(\chi_T, \chi_S) := \frac{1}{|G|} \sum_{\sigma \in G} \chi_T(\sigma)\overline{\chi_S(\sigma)}$$

is always a real number.

2. Denote by $\mathbb{C}S$ the vector space of all complex valued functions on a finite set S. Let G be a finite group acting on S by permutations. The *permutation representation* of G on S is defined by $T(\sigma)f = f \circ \sigma^{-1}$ for $f \in \mathbb{C}S$ and $\sigma \in G$. Let χ_T denote the character of this representation. Show that

$$\begin{aligned} \chi_T(\sigma) &= \#\{s \in S : \sigma s = s\} \\ &= \text{ number of fixed points of } \sigma \text{ in } S. \end{aligned}$$

3. (Burnside's orbit formula) If a finite group G acts on a finite set S, with χ being the character of the permutation representation, then the number of G-orbits in S is

$$\begin{aligned} \text{number of } G\text{-orbits in } S &= \frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma) \\ &= (\chi, 1_G). \end{aligned}$$

where 1_G denotes the one-dimensional trivial representation and the inner product is as defined in problem 1.

4. Let T be a linear representation of a finite group G with character χ. Show the trivial 1-dimensional representation appears in T with multiplicity $\frac{1}{|G|} \sum_{\sigma \in G} \chi(\sigma)$.

5. (Fourier series on $\mathbb{Z}/(m)$) Representing by $x = k/m$ (modulo 1), $k = 0, 1, \ldots, m-1$, the elements of $G = \mathbb{Z}/(m)$, define the 1-dimensional representations

$$T_l(x)z := e^{2\pi i l x} z,$$

for $x \in G$, $z \in \mathbb{C}$ and $l = 0, 1, \ldots, m-1$. Show that the respective characters, χ_l, $l = 0, \ldots, m-1$, constitute an orthonormal basis for the space $\mathbb{C}G$ with respect to the inner product defined in problem 1. (We think of $\mathbb{C}G$ as the space of 1-periodic functions on $(1/m)\mathbb{Z}$.)