Math 5051 - Homework 1

Due 9/11/08

1. (Problem 5, page 24) If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: Show that the latter object is a σ-algebra.)

2. (Problem 12, page 27) Let $(X, \mathcal{M}, \mu)$ be a finite measure space.
   (a) If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, the $\mu(E) = \mu(F)$.
   (b) Say that $E \sim F$ if $\mu(E \triangle F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.
   (c) For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.

3. (Problem 17, page 32) If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

4. (Problem 18, page 32) Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.
   (a) For any $E \subset X$ and $\epsilon > 0$ there exist $A \in \mathcal{A}_\sigma$ with $E \subset A$ and $\mu^*(A) \leq \mu^*(E) + \epsilon$.
   (b) If $\mu^*(E) < \infty$, then E is μ^*-measurable iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subset B$ and $\mu^*(B \setminus E) = 0$.
   (c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

5. (Poincaré recurrence) Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$. Let $T : X \to X$ be a *measurable*, *measure preserving* map. (T is measurable if $T^{-1}(E) \in \mathcal{M}$ for all $E \in \mathcal{E}$; and it is *measure preserving* if $\mu(T^{-1}(E)) = \mu(E)$ for all $E \in \mathcal{M}$.) If $E \in \mathcal{M}$ is such that $\mu(E) > 0$, then almost every point of E returns to E infinitely often under the positive iterations of T. In other words, there exists an $\mathcal{M}$-measurable set $F \subset E$, $\mu(F) = \mu(E)$, such that for each $x \in F$ there is a sequence $n_1 < n_2 < \ldots$ of natural numbers with $T^{n_i}(x) \in F$ for each i. (Hint: Look up any book on *Ergodic Theory*.)