Math 5051 - Homework 11

Due 11/20/08

1. (Problem 31, page 108) Let \( F(x) = x^2 \sin(x^{-1}) \) and \( G(x) = x^2 \sin(x^{-2}) \) for \( x \neq 0 \), and \( F(0) = G(0) = 0 \).

   (a) \( F \) and \( G \) are differentiable everywhere (including 0).

   (b) \( F \in BV([-1,1]) \), but \( G \notin BV([-1,1]) \).

2. (Problem 35, page 108) If \( F \) and \( G \) are absolutely continuous on \([a, b]\), then so is \( FG \), and

\[
\int_a^b (FG' + GF')(x) \, dx = F(b)G(b) - F(a)G(a)
\]

3. (Problem 37, page 108) \( F : \mathbb{R} \rightarrow \mathbb{C} \) is said to be **Lipschitz continuous** if there is a constant \( M \) such that \( |F(x) - F(y)| \leq M|x-y| \) for all \( x, y \in \mathbb{R} \). Show that \( F \) is Lipschitz continuous iff \( F \) is absolutely continuous and \( |F'| \leq M \) a.e.

4. (Problem 42, page 109) A function \( F : (a, b) \rightarrow \mathbb{R} \) \((\infty \leq a < b \leq \infty)\) is called **convex** if

\[
F(\lambda s + (1-\lambda)t) \leq \lambda F(s) + (1-\lambda)F(t)
\]

for all \( s, t \in (a, b) \) and \( \lambda \in (0, 1) \). (Geometrically, this says that the graph of \( F \) over the interval from \( s \) to \( t \) lies underneath the line segment joining \((s, F(s))\) to \((t, F(t))\).)

   (a) \( F \) is convex iff for all \( s, t, s', t' \in (a, b) \) such that \( s \leq s' < t' \) and \( s < t \leq t' \),

\[
\frac{F(t) - F(s)}{t-s} \leq \frac{F(t') - F(s')}{t'-s'}.
\]

   (b) \( F \) is convex iff \( F \) is absolutely continuous on every compact subinterval of \((a, b)\) and \( F' \) is increasing (on the set where it is defined).

   (c) If \( F \) is convex and \( t_0 \in (a, b) \), there exists \( \beta \in \mathbb{R} \) such that \( F(t) - F(t_0) \geq \beta(t-t_0) \) for all \( t \in (a, b) \).

   (d) (Jensen’s inequality) If \((X, \mathcal{M}, \mu)\) is a measure space with \( \mu(X) = 1 \), \( g : X \rightarrow (a, b) \) is in \( L^1(\mu) \), and \( F \) is convex on \((a, b)\), then

\[
F\left(\int g \, d\mu\right) \leq \int F \circ g \, d\mu.
\]

   (Let \( t_0 = \int g \, d\mu \) and \( t = g(x) \) in (c), and integrate.)