

Math 5051 - Homework 11

Due 11/20/08

- (Problem 31, page 108) Let $F(x) = x^2 \sin(x^{-1})$ and $G(x) = x^2 \sin(x^{-2})$ for $x \neq 0$, and $F(0) = G(0) = 0$.
 - F and G are differentiable everywhere (including 0).
 - $F \in BV([-1, 1])$, but $G \notin BV([-1, 1])$.

- (Problem 35, page 108) If F and G are absolutely continuous on $[a, b]$, then so is FG , and

$$\int_a^b (FG' + GF')(x) dx = F(b)G(b) - F(a)G(a)$$

- (Problem 37, page 108) $F : \mathbb{R} \rightarrow \mathbb{C}$ is said to be **Lipschitz continuous** if there is a constant M such that $|F(x) - F(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Show that F is Lipschitz continuous iff F is absolutely continuous and $|F'| \leq M$ a.e.
- (Problem 42, page 109) A function $F : (a, b) \rightarrow \mathbb{R}$ ($-\infty \leq a < b \leq \infty$) is called **convex** if

$$F(\lambda s + (1 - \lambda)t) \leq \lambda F(s) + (1 - \lambda)F(t)$$

for all $s, t \in (a, b)$ and $\lambda \in (0, 1)$. (Geometrically, this says that the graph of F over the interval from s to t lies underneath the line segment joining $(s, F(s))$ to $(t, F(t))$.)

- F is convex iff for all $s, t, s', t' \in (a, b)$ such that $s \leq s' < t'$ and $s < t \leq t'$,

$$\frac{F(t) - F(s)}{t - s} \leq \frac{F(t') - F(s')}{t' - s'}.$$

- F is convex iff F is absolutely continuous on every compact subinterval of (a, b) and F' is increasing (on the set where it is defined).
- If F is convex and $t_0 \in (a, b)$, there exists $\beta \in \mathbb{R}$ such that $F(t) - F(t_0) \geq \beta(t - t_0)$ for all $t \in (a, b)$.
- (**Jensen's inequality**) If (X, \mathcal{M}, μ) is a measure space with $\mu(X) = 1$, $g : X \rightarrow (a, b)$ is in $L^1(\mu)$, and F is convex on (a, b) , then

$$F\left(\int g d\mu\right) \leq \int F \circ g d\mu.$$

(Let $t_0 = \int g d\mu$ and $t = g(x)$ in (c), and integrate.)