Math 5051 - Homework 12 (The last one!)

Due 12/16/08

1. (Problem 32, page 127) A topological space X is Hausdorff iff every net in X converges to at most one point. (If X is not Hausdorff, let x and y be distinct points with no disjoint neighborhoods, and consider the directed set $\mathcal{N}_x \times \mathcal{N}_y$ where $\mathcal{N}_x, \mathcal{N}_y$ are the families of neighborhoods of x, y.)

2. (Problem 38, page 130) Suppose that $(X, \mathcal{T})$ is a compact Hausdorff space and $\mathcal{T}'$ is another topology on X. If $\mathcal{T}'$ is strictly stronger than $\mathcal{T}$, then $(X, \mathcal{T}')$ is Hausdorff but not compact. If $\mathcal{T}'$ is strictly weaker than $\mathcal{T}$, than $(X, \mathcal{T}')$ is compact but not Hausdorff.

3. (Problem 51, page 135) If X and Y are topological spaces, $\phi \in C(X, Y)$ is called **proper** if $\phi^{-1}(K)$ is compact in X for every compact $K \subset Y$. Suppose that X and Y are LCH spaces and X^* and Y^* are their one-point compactifications. If $\phi \in C(X, Y)$, then ϕ is proper iff ϕ extends continuously to a map from X^* to Y^* by setting $\phi(\infty_X) = \infty_Y$.

4. (Problem 52, page 135) The one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

5. (Not for credit) Read section 4.6, pages 136-138.

6. (Problem 63, page 138) Let $K \in C([0,1] \times [0,1])$. For $f \in C([0,1])$, let

$$Tf(x) = \int_0^1 K(x,y) f(y)\, dy.$$

Then $Tf \in C([0,1])$, and $\{Tf : \|f\|_u \leq\}$ is precompact in $C([0,1])$.

7. (Problem 65, page 138) Let (X, ρ) be a metric space. A function $f \in C(X)$ is called **Hölder continuous of exponent** α ($\alpha > 0$) if the quantity

$$N_\alpha(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x,y)^\alpha}$$

is finite. If X is compact, $\{f \in C(X) : \|f\|_u \leq 1 \text{ and } N_\alpha(f) \leq 1\}$ is compact in $C(X)$.