Math 5051 - Homework 6

Midterm special (6 questions!)

Due 10/16/08

1. (Problem 46, page 68) Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

2. (Problem 48, page 69) Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n+1$, and $f(m,n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\iint f \, d\mu \, d\nu$ and $\iint f \, d\nu \, d\mu$ exist and are unequal.

3. (Problem 49, page 69) Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

   (a) If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

   (b) If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of mu and ν is needed.)

4. (problem 51, page 69) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be arbitrary measure spaces (not necessarily σ-finite).

   (a) If $f : X \to \mathbb{C}$ is $\mathcal{M}$-measurable, $g : Y \to \mathbb{C}$ is $\mathcal{N}$-measurable, and $h(x,y) = f(x)g(y)$, then h is $\mathcal{M} \otimes \mathcal{N}$-measurable.

   (b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = [\int f \, d\mu][\int g \, d\nu]$.

5. (Problem 52, page 69) The Fubini-Tonelli theorem is valid when $(X, \mathcal{M}, \mu)$ is an arbitrary measure space and Y is a countable set, $\mathcal{N} = \mathcal{P}(\mathbb{N})$, and ν is counting measure on Y. (Cf. Theorems 2.15 and 2.25.)

6. (Problem 54, page 77) How much of Theorem 2.44 remains valid if T is not invertible?