Math 5051 - Homework 6

Midterm special (6 questions!)

Due 10/16/08

1. (Problem 46, page 68) Let $X = Y = [0,1]$, $M = N = B_{[0,1]}$, $\mu$ = Lebesgue measure, and $\nu$ = counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\iint \chi_D \, d\mu \, d\nu$, $\iint \chi_D \, d\nu \, d\mu$, and $\int \chi_D \, d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D \, d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

2. (Problem 48, page 69) Let $X = Y = \mathbb{N}$, $M = N = \mathcal{P}(\mathbb{N})$, $\mu = \nu$ = counting measure. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n + 1$, and $f(m,n) = 0$ otherwise. Then $\int |f| \, d(\mu \times \nu) = \infty$, and $\iint f \, d\mu \, d\nu$ and $\iint f \, d\nu \, d\mu$ exist and are unequal.

3. (Problem 49, page 69) Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

(a) If $E \in M \times N$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. $x$ and $y$.

(b) If $f$ is $\mathcal{L}$-measurable and $f = 0$ $\lambda$-a.e., then $f_x$ and $f^y$ are integrable for a.e. $x$ and $y$, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. $x$ and $y$. (Here the completeness of $\mu u$ and $\nu$ is needed.)

4. (Problem 51, page 69) Let $(X,M,\mu)$ and $(Y,N,\nu)$ be arbitrary measure spaces (not necessarily $\sigma$-finite).

(a) If $f : X \to \mathbb{C}$ is $M$-measurable, $g : Y \to \mathbb{C}$ is $N$-measurable, and $h(x,y) = f(x)g(y)$, then $h$ is $M \otimes N$-measurable.

(b) If $f \in L^1(\mu)$ and $g \in L^1(\nu)$, then $h \in L^1(\mu \times \nu)$ and $\int h \, d(\mu \times \nu) = [\int f \, d\mu][\int g \, d\nu]$.

5. (Problem 52, page 69) The Fubini-Tonelli theorem is valid when $(X,M,\mu)$ is an arbitrary measure space and $Y$ is a countable set, $N = \mathcal{P}(\mathbb{N})$, and $\nu$ is counting measure on $Y$. (Cf. Theorems 2.15 and 2.25.)

6. (Problem 54, page 77) How much of Theorem 2.44 remains valid if $T$ is not invertible?