Math 5051 - Homework 7

Due 10/23/08

1. (Problem 55, page 77) Investigate the existence and equality of $\int_E f\,dm^2$, $\int_0^1\int_0^1 f(x,y)\,dx\,dy$, and $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ for the following f.
   (a) $f(x,y) = (x^2-y^2)(x^2+y^2)^{-2}$.
   (b) $f(x,y) = (1-xy)^{-a}$ $(a>0)$.
   (c) $f(x,y) = (x-\frac{1}{2})^{-3}$ if $0<y<|x-\frac{1}{2}|$, $f(x,y)=0$ otherwise.

2. (Problem 56, page 77) If f is Lebesgue integrable on $(0,a)$ and $g(x) = \int_x^a t^{-1}f(t)\,dt$, then g is integrable on $(0,a)$ and $\int_0^a g(x)\,dx = \int_0^a f(x)\,dx$.

3. (Problem 58, page 77) Show that $\int_0^\infty e^{-sx}x^{-1}\sin x\,dx = \frac{1}{4}\log(1+4s^{-2})$ for $s>0$ by integrating $e^{-sx}\sin 2xy$ with respect to x and y.

4. (Problem 65, page 80) Define $G:\mathbb{R}^n\to\mathbb{R}^n$ by $G(r,\phi_1,\dots,\phi_{n-2},\theta) = (x_1,\dots,x_n)$, where

$$
\begin{aligned}
x_1 &= r\cos\phi_1\\
x_2 &= r\sin\phi_1\cos\phi_2\\
x_3 &= r\sin\phi_1\sin\phi_2\cos\phi_3\\
&\cdots\\
x_{n-1} &= r\sin\phi_1\cdots\sin\phi_{n-2}\cos\theta\\
x_n &= r\sin\phi_1\cdots\sin\phi_{n-2}\sin\theta.
\end{aligned}
$$

   (a) G maps $\mathbb{R}^n$ onto $\mathbb{R}^n$, and $|G(r,\phi_1,\dots,\phi_{n-2},\theta)| = |r|$.
   (b) $\det D_{(r,\phi_1,\dots,\phi_{n-2},\theta)}G = r^{n-1}\sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}$.
   (c) Let $\Omega = (0,\infty)\times(0,\pi)^{n-2}\times(0,2\pi)$. Then $G|_\Omega$ is a diffeomorphism and $m(\mathbb{R}^n\setminus G(\Omega)) = 0$.
   (d) Let $F(\phi_1,\dots,\phi_{n-2},\theta) = G(1,\phi_1,\dots,\phi_{n-2},\theta)$ and $\Omega' = (0,\pi)^{n-2}\times(0,2\pi)$. Then $(F|_{\Omega'})^{-1}$ defines a coordinate system on S^{n-1} except on a σ-null set, and the measure σ is given in these coordinates by

$$d\sigma(\phi_1,\dots,\phi_{n-2},\theta) = \sin^{n-2}\phi_1\sin^{n-3}\phi_2\cdots\sin\phi_{n-2}\,d\phi_1\cdots d\phi_{n-2}.$$