Math 5052 - Homework 1

Due 1/22/09

1. (Problem 66, page 142) Let $1 - \sum_1^\infty c_n t^n$ be the Maclaurin series for $(1-t)^{1/2}$.

   (a) The series converges absolutely and uniformly on compact subsets of $(-1, 1)$, as does the termwise differentiated series $-\sum_1^\infty n c_n t^{n-1}$. Thus, if $f(t) = 1 - \sum_1^\infty c_n t^n$, then $f'(t) = -\sum_1^\infty n c_n t^{n-1}$.

   (b) By explicit calculation, $f(t) = -2(1-t)f'(t)$, from which it follows that $(1-t)^{-1/2} f(t)$ is constant. Since $f(0) = 1$, $f(t) = (1-t)^{1/2}$.

2. (Problem 68, page 142) Let X and Y be compact Hausdorff spaces. The algebra generated by functions of the form $f(x, y) = g(x)h(y)$, where $g \in C(X)$ and $h \in C(Y)$, is dense in $C(X \times Y)$.

3. (Problem 70, page 142) Let X be a compact Hausdorff space. An **ideal** in $C(X, \mathbb{R})$ is a subalgebra $\mathcal{I}$ of $C(X, \mathbb{R})$ such that if $f \in \mathcal{I}$ and $g \in C(X, \mathbb{R})$ then $fg \in \mathcal{I}$.

   (a) If $\mathcal{I}$ is an ideal in $C(X, \mathbb{R})$, let $h(\mathcal{I}) = \{x \in X : f(x) = 0 \text{ for all } f \in \mathcal{I}\}$. Then $h(\mathcal{I})$ is a closed subset of X, called the **hull** of $\mathcal{I}$.

   (b) If $E \subset X$, let $k(E) = \{f \in C(X, \mathbb{R}) : f(x) = 0 \text{ for all } x \in E\}$. Then $k(E)$ is a closed ideal in $C(X, \mathbb{R})$, called the **kernel** of E.

   (c) If $E \subset X$, then $h(k(E)) = \overline{E}$.

   (d) If $\mathcal{I}$ is an ideal in $C(X, \mathbb{R})$, then $k(h(\mathcal{I})) = \overline{\mathcal{I}}$. (Hint: $k(h(\mathcal{I}))$ may be identified with a subalgebra of $C_0(U, \mathbb{R})$ where $U = X \setminus h(\mathcal{I})$.)

   (e) The closed subsets of X are in one-to-one correspondence with the closed ideals of $C(X, \mathbb{R})$.

4. (Problem 74, page 146) Consider $\mathbb{N}$ (with the discrete topology) as a subset of its Stone-Čech compactification $\beta\mathbb{N}$.

   (a) If A and B are disjoint subsets of $\mathbb{N}$, their closures in $\beta\mathbb{N}$ are disjoint. (Hint: $\chi_A \in C(\mathbb{N}, I)$.)

   (b) No sequence in $\mathbb{N}$ converges in $\beta\mathbb{N}$ unless it is eventually constant (son $\beta\mathbb{N}$ is emphatically *not* sequencially compact.)

5. (Problem 75, page 146) Suppose X is a completely regular space. The set M of nonzero algebra homomorphisms from $BC(X, \mathbb{R})$ to $\mathbb{R}$, equipped with the topology of pointwise convergence, is homeomorphic to βX. (See Exercise 71. This realization of βX is the natural one from the point of view of Banach algebra theory.)