Math 5052 - Homework 2

Due 1/29/09

1. (Problem 7, page 155) Let $\mathcal{X}$ be a Banach space.

   (a) If $T \in L(\mathcal{X}, \mathcal{X})$ and $\|I - T\| < 1$ where I is the identity operator, then T is invertible; in fact, the series $\sum_0^\infty (I - T)^n$ converges in $L(\mathcal{X}, \mathcal{X})$ to T^{-1}.

   (b) If $T \in L(\mathcal{X}, \mathcal{X})$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then S is invertible. Thus the set of invertible operators is open in $L(\mathcal{X}, \mathcal{X})$.

2. (Problem 8, page 155) Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space. (Use Theorem 5.1.)

3. (Problem 15, page 156) Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are normed vector spaces and $T \in L(\mathcal{X}, \mathcal{Y})$. Let $\mathcal{N}(T) = \{x \in \mathcal{X} : Tx = 0\}$.

   (a) $\mathcal{N}(T)$ is a closed subspace of $\mathcal{X}$.

   (b) There is a unique $S \in L(\mathcal{X}/\mathcal{N}(T), \mathcal{Y})$ such that $T = S \circ \pi$ where $\pi : \mathcal{X} \to \mathcal{X}/\mathcal{N}(T)$ is the projection (see Exercise 12). Moreover, $\|S\| = \|T\|$.

4. (Problem 19, page 160) Let $\mathcal{X}$ be an infinite-dimensional normed vector space.

   (a) There is a sequence $\{x_j\}$ in $\mathcal{X}$ such that $\|x_j\| = 1$ and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. (Construct x_j inductively, using Exercises 12b and 18.)

   (b) $\mathcal{X}$ is not locally compact.

5. (Problem 21, page 160) If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.