Math 5052 - Homework 2

Due 1/29/09

1. (Problem 7, page 155) Let $X$ be a Banach space.

(a) If $T \in L(X, X)$ and $\|I - T\| < 1$ where $I$ is the identity operator, then $T$ is invertible; in fact, the series $\sum_0^{\infty} (I - T)^n$ converges in $L(X, X)$ to $T^{-1}$.

(b) If $T \in L(X, X)$ is invertible and $\|S - T\| < \|T^{-1}\|^{-1}$, then $S$ is invertible. Thus the set of invertible operators is open in $L(X, X)$.

2. (Problem 8, page 155) Let $(X, M)$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, M)$. Then $\|\mu\| = |\mu|(X)$ is a norm on $M(X)$ that makes $M(X)$ into a Banach space. (Use Theorem 5.1.)

3. (Problem 15, page 156) Suppose that $X$ and $Y$ are normed vector spaces and $T \in L(X, Y)$. Let $N(T) = \{x \in X : Tx = 0\}$.

(a) $N(T)$ is a closed subspace of $X$.

(b) There is a unique $S \in L(X/N(T), Y)$ such that $T = S \circ \pi$ where $\pi : X \to X/N(T)$ is the projection (see Exercise 12). Moreover, $\|S\| = \|T\|$.

4. (Problem 19, page 160) Let $X$ be an infinite-dimensional normed vector space.

(a) There is a sequence $\{x_j\}$ in $X$ such that $\|x_j\| = 1$ and $\|x_j - x_k\| \geq \frac{1}{2}$ for $j \neq k$. (Construct $x_j$ inductively, using Exercises 12b and 18.)

(b) $X$ is not locally compact.

5. (Problem 21, page 160) If $M$ is a finite-dimensional subspace of a normed vector space $X$, there is a closed subspace $N$ such that $M \cap N = \{0\}$ and $M + N = X$. 