Math 5052 - Homework 3

Due 2/05/09

1. (Problem 27, page 164) There exist meager subsets of $\mathbb{R}$ whose complements have Lebesgue measure zero.

2. (Problem 30, page 164) Let $\mathcal{Y} = C([0,1])$ and $\mathcal{X} = C^1([0,1])$, both equipped with the uniform norm.

   (a) $\mathcal{X}$ is not complete.

   (b) The map $(d/dx) : \mathcal{X} \to \mathcal{Y}$ is closed (see Exercise 9) but not bounded.

3. (Problem 32, page 164) Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on the vector space $\mathcal{X}$ such that $\|\cdot\|_1 \leq \|\cdot\|_2$. If $\mathcal{X}$ is complete with respect to both norms, then the norms are equivalent.

4. (Problem 36, page 164) Let $\mathcal{X}$ be a separable Banach space space and let μ be counting measure on $\mathbb{N}$. Suppose that $\|x_n\|_1^\infty$ is a countable dense subset of the unit ball of $\mathcal{X}$, and define $T : L^1(\mu) \to \mathcal{X}$ by $Tf = \sum_1^\infty f(n)x_n$.

   (a) T is bounded.

   (b) T is surjective.

   (c) $\mathcal{X}$ is isomorphic to a quotient space of $L^1(\mu)$. (Use Exercise 35.)

5. (Problem 45, page 170) The space $C^\infty(\mathbb{R})$ of all infinitely differentiable functions on $\mathbb{R}$ has a Fréchet space topology with respect to which $f_n \to f$ iff $f_n^{(k)} \to f^{(k)}$ uniformly on compact sets for all $k \geq 0$.