Math 5052 - Homework 4

Due 2/12/09

(It's not necessary to write up solutions to the many exercises referenced in the ones assigned below.)

1. (Problem 49, page 170) Suppose that $\mathcal{X}$ is an infinite-dimensional Banach space.
   (a) Every nonempty weakly open set in $\mathcal{X}$, and every nonempty weak*-open set in $\mathcal{X}^*$, is unbounded (with respect to the norm).
   (b) Every bounded subset of $\mathcal{X}$ is nowhere dense in the weak topoloty, and every bounded subset of $\mathcal{X}^*$ is nowhere dense in the weak* topology. (Use Exercise 48b,c.)
   (c) $\mathcal{X}$ is meager in itself with respect to the weak topology, and $\mathcal{X}^*$ is meager in itself with respect to the weak* topology.
   (d) The weak* topology on $\mathcal{X}^*$ is not defined by any translation-invariant metric. (Use Exercise 48d.)

2. (Problem 54, page 177) For any nonempty set A, $l^2(A)$ is complete.

3. (Problem 57, page 177) Suppose that $\mathcal{H}$ is a Hilbert space and $T \in L(\mathcal{H}, \mathcal{H})$.
   (a) There is a unique $T^* \in L(\mathcal{H}, \mathcal{H})$, called the **adjoint** of T, such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in \mathcal{H}$. (Cf. Exercise 22. We have $T^* = V^{-1}T^\dagger V$ where V is the conjugate-linear isomorphism from $\mathcal{H}$ to $\mathcal{H}^*$ in Theorem 5.25, $(Vy)(x) = \langle x, y\rangle$.)
   (b) $\|T^*\| = \|T\|$, $\|T^*T\| = \|T\|^2$, $(aS + bT)^* = \overline{a}S^* + \overline{b}T^*$, $(ST)^* = T^*S^*$, and $T^{**} = T$.
   (c) Let $\mathcal{R}$ and $\mathcal{N}$ denote range and nullspace; then $\mathcal{R}(T)^\perp = \mathcal{N}(T^*)$ and $\mathcal{N}(T)^\perp = \overline{\mathcal{R}(T^*)}$.
   (d) T is unitary iff T is invertible and $T^{-1} = T^*$.

4. (Problem 62, page 178) In this exercise the measure defining the L^2 spaces is Lebesgue measure.
   (a) $C([0, 1])$ is dense in $L^2([0, 1])$. (Adapt the proof of Theorem 2.26.)
   (b) The set of polynomials is dense in $L^2([0, 1])$.
   (c) $L^2([0, 1])$ is separable.
   (d) $L^2(\mathbb{R})$ is separable. (Use Exercise 60.)
   (e) $L^2(\mathbb{R}^n)$ is separable. (Use Exercise 61.)

5. (Problem 63, page 178) Let $\mathcal{H}$ be an infinite dimensional Hilbert space.
   (a) Every orthonormal sequence in $\mathcal{H}$ converges weakly to 0.
   (b) The unit sphere $S = \{x : \|x\| = 1\}$ is weakly dense in the unit ball $B = \{x : \|x\| \leq 1\}$. (In fact, every $x \in B$ is the weak limit of a sequence in S.)