Math 5052 - Homework 5

Due 2/19/09

1. (Problem 67, page 178, **The Mean Ergodic Theorem**). Let U be a unitary operator on the Hilbert space $\mathcal{H}$, $\mathcal{M} = \{x : Ux = x\}$, P the orthogonal projection onto $\mathcal{M}$ (Exercise 58), and $S_n = n^{-1}\sum_0^{n-1} U^j$. Then $S_n \to P$ in the strong operator topology. (If $x \in \mathcal{M}$, then $S_n x = x$; if $x = y - Uy$ for some y, then $S_n x \to 0$. By Exercise 57d, $\mathcal{M} = \{x : U^* x = x\}$. Apply Exercise 57c with $T = I - U$.)

2. (Problem 5, page 186) Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily large finite measure. (For the "if" implication: In the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \le \mu(E_n) < \infty$. Consider $f = \sum a_n \chi_{E_n}$ for suitable constants a_n.) What about the case $q = \infty$?

3. (Problem 6, page 187) Suppose $0 < p_0 < p_1 \le \infty$. Find examples of functions f on $(0, \infty)$ (with Lebesgue measure), such that $f \in L^p$ iff (a) $p_0 < p < p_1$, (b) $p_0 \le p \le p_1$, (c) $p = p_0$. (Consider functions of the form $f(x) = x^{-a}|\log x|^b$.)

4. (Problem 7, page 187) If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q\to\infty} \|f\|_q$.

5. (Problem 10, page 187) Suppose $1 \le p < \infty$, If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\| \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in §2.3.)