Math 5052 - Homework 5

Due 2/19/09

1. (Problem 67, page 178, The Mean Ergodic Theorem). Let $U$ be a unitary operator on the Hilbert space $H$, $M = \{x : Ux = x\}$, $P$ the orthogonal projection onto $M$ (Exercise 58), and $S_n = n^{-1} \sum_{j=0}^{n-1} U^j$. Then $S_n \to P$ in the strong operator topology. (If $x \in M$, then $S_n x = x$; if $x = y - Uy$ for some $y$, then $S_n x \to 0$. By Exercise 57d, $M = \{x : U^* x = x\}$. Apply Exercise 57c with $T = I - U$.)

2. (Problem 5, page 186) Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff $X$ contains sets of arbitrarily large finite measure. (For the “if” implication: In the first case there is a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < 2^{-n}$, and in the second case there is a disjoint sequence $\{E_n\}$ with $1 \leq \mu(E_n) < \infty$. Consider $f = \sum a_n \chi_{E_n}$ for suitable constants $a_n$.) What about the case $q = \infty$?

3. (Problem 6, page 187) Suppose $0 < p_0 < p_1 \leq \infty$. Find examples of functions $f$ on $(0, \infty)$ (with Lebesgue measure), such that $f \in L^p$ iff (a) $p_0 < p < p_1$, (b) $p_0 \leq p \leq p_1$, (c) $p = p_0$. (Consider functions of the form $f(x) = x^{-a} |\log x|^b$.)

4. (Problem 7, page 187) If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $\|f\|_\infty = \lim_{q \to \infty} \|f\|_q$.

5. (Problem 10, page 187) Suppose $1 \leq p < \infty$, If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\| \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in §2.3.)