Math 5052 - Homework 6

Due 2/26/09

1. (Problem 12, page 187) If $p \neq 2$, the L^p norm does not arise from an inner product on L^p, except in trivial cases when $\dim(L^p) \leq 1$. (Show that the parallelogram law fails.)

2. (Problem 13, page 187) $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable. (There is an uncountable set $\mathcal{F} \subset L^\infty$ such that $\|f - g\|_\infty \geq 1$ for all $f, g \in \mathcal{F}$ with $f \neq g$.)

3. (Problem 14, page 187) If $g \in L^\infty$, the operator T defined by $Tf = fg$ is bounded on L^p for $1 \leq p \leq \infty$. Its operator norm is at most $\|g\|_\infty$, with equality if μ is semifinite.

4. (Problem 18, page 191) The self-duality of L^2 follows from Hilbert space theory (Theorem 5.25), and this fact can be used to prove the Lebesgue-Radon-Nikodym theorem by the following argument due to von Neumann. Suppose that μ, ν are positive finite measures on $(X, \mathcal{M})$ (the σ-finite case follows easily as in §3.2), and let $\lambda = \mu + \nu$.

   (a) The map $f \mapsto \int f\, d\nu$ is a bounded linear functional on $L^2(\lambda)$, so $\int f\, d\nu = \int fg\, d\lambda$ for some $g \in L^2(\lambda)$. Equivalently, $\int f(1-g)\, d\nu = \int fg\, d\mu$ for $f \in L^2(\lambda)$.

   (b) $0 \leq g \leq 1$ λ-a.e., so we may assume $0 \leq g \leq 1$ everywhere.

   (c) Let $A = \{x : g(x) < 1\}$, $B = \{x : g(x) = 1\}$, and set $\nu_a(E) = \nu(A \cap E)$, $\nu_s(E) = \nu(B \cap E)$. Then $\nu_s \perp \mu$ and $\nu_a \ll \mu$; in fact, $d\nu_a = g(1-g)^{-1}\chi_A d\mu$.

5. (Problem 21, page 192) If $1 < p < \infty$, $f_n \to f$ weakly in $l^p(A)$ iff $\sup_n \|f\|_p < \infty$ and $f_n \to f$ pointwise.