

Math 5052 - Homework 7

Due 03/05/09

1. (Problem 27, page 196. **Hilbert's Inequality**) The operator $Tf(x) = \int_0^\infty (x+y)^{-1}f(y) dy$ satisfies $\|Tf\|_p \leq C_p\|f\|_p$ for $1 < p < \infty$, where $C_p = \int_0^\infty x^{-1/p}(x+y)^{-1}dx$. (For those who know about contour integrals: Show that $C_p = \pi \csc(\pi/p)$.)
2. (Problem 31, page 196. **A Generalized Hölder Inequality**) Suppose that $1 \leq p_j \leq \infty$ and $\sum_1^n p_j^{-1} = r^{-1} \leq 1$. If $f_j \in L^{p_j}$ for $j = 1, \dots, n$, then $\prod_1^n f_j \in L^r$ and $\|\prod_1^n f_j\|_r \leq \prod_1^n \|f_j\|_{p_j}$. (First do the case $n = 2$.)
3. (Problem 32, page 197) Suppose that (X, \mathcal{M}, μ) and (Y, \mathcal{N}, ν) are σ -finite measure spaces and $K \in L^2(\mu \times \nu)$. If $f \in L^2(\nu)$, the integral $Tf(x) = \int K(x, y)f(y) d\nu(y)$ converges absolutely for a.e. $x \in X$; moreover, $Tf \in L^2(\mu)$ and $\|Tf\|_2 \leq \|K\|_2\|f\|_2$.
4. (Problem 36, page 199) If $f \in \text{weak } L^p$ and $\mu(\{x : f(x) \neq 0\}) < \infty$, then $f \in L^q$ for all $q < p$. On the other hand, if $f \in (\text{weak } L^p) \cap L^\infty$, then $f \in L^q$ for all $q > p$.
5. (Problem 39, page 199) If $f \in L^p$, then $\lim_{\alpha \rightarrow 0} \alpha^p \lambda_f(\alpha) = \lim_{\alpha \rightarrow \infty} \alpha^p \lambda_f(\alpha) = 0$. (First suppose f is simple.)