Math 5052 - Homework 8

Due 03/26/09

1. (Problem 1, page 215) Let X be a LCH space, Y a closed subset of X (which is an LCH space in the relative topology), and μ a Radon measure on Y. Then $I(f) = \int (f|Y)\, d\mu$ is a positive linear functional on $C_c(X)$, and the induced Radon measure ν on X is given by $\nu(E) = \mu(E \cap Y)$.

2. (Problem 2, page 215) Let μ be a Radon measure on X.

   (a) Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open and $\mu(U) = 0$. The complement of N is called the **support** of μ.

   (b) $x \in \text{supp}(\mu)$ iff $\int f\, d\mu > 0$ for every $f \in C_c(X, [0,1])$ such that $f(x) > 0$.

3. (Problem 8, page 220) Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \geq 0$, then $\nu(E) = \int_E \phi\, d\mu$ is a Radon measure. (Use Corollary 3.6.)

4. (Problem 18, page 224) If μ is a σ-finite Radon measure on X and $\nu \in \mathcal{M}(X)$, let $\nu = \nu_1 + \nu_2$ be the Lebesgue decomposition of ν with respect to μ. Then ν_1 and ν_2 are Radon. (Use Exercise 8.)

5. (Problem 25, page 225) Let μ be a Radon measure on X such that every nonempty open set has positive measure (e.g., Lebesgue measure). For each $x \in X$ there is a net $\{f_\alpha\}$ in $L^1(\mu)$ that converges vaguely in $\mathcal{M}(X)$ to the point mass at x. If X is first countable, the net can be taken to be a sequence. (Consider functions of the form $\mu(U)^{-1}\chi_U$.)