Math 5052 - Homework 10

Due 04/09/09

1. (Problem 2, page 239) Observe that the binomial theorem can be written as follows:

$$(x_1 + x_2)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha \qquad (x = (x_1, x_2), \alpha = (\alpha_1, \alpha_2)).$$

   Prove the following generalizations:

   (a) The multinomial theorem: If $x \in \mathbb{R}^n$,

$$(x_1 + \cdots + x_n)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} x^\alpha.$$

   (b) The n-dimensional binomial theorem:

$$(x + y)^\alpha = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} x^\beta y^\gamma.$$

2. (Problem 3, page 239) Let $\eta(t) = e^{-1/t}$ for $t > 0$, $\eta(t) = 0$ for $t \le 0$.

   (a) For $k \in \mathbb{N}$ and $t > 0$, $\eta^{(k)}(t) = P_k(1/t)e^{-1/t}$, where P_k is a polynomial of degree $2k$.

   (b) $\eta^{(k)}(0)$ exists and equals zero for all $k \in \mathbb{N}$.

3. (Problem 4, page 239) If $f \in L^\infty$ and $\|\tau_y f - f\|_\infty \to 0$ as $y \to 0$, then f agrees a.e. with a uniformly continuous function. (Let $A_r f$ be as in Theorem 3.18. Then $A_r f$ is uniformly continuous for $r > 0$ and uniformly Cauchy as $r \to 0$.)

4. (Problem 7, page 246) If f is locally integrable on $\mathbb{R}^n$ and $g \in C^k$ has compact support, then $f * g \in C^k$.

5. (Problem 8, page 246) Suppose that $f \in L^p(\mathbb{R})$. If there exists $h \in L^p(\mathbb{R})$ such that

$$\lim_{y \to 0} \|y^{-1}(\tau_{-y} f - f) - h\|_p = 0,$$

   we call h the (**strong**) L^p **derivative** of f. If $f \in L^p(\mathbb{R}^n)$, L^p partial derivatives of f are defined similarly. Suppose that p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and the L^p derivative $\partial_j f$ exists. Then $\partial_j(f * g)$ exists (in the ordinary sense) and equals $(\partial_j f) * g$.