Math 5052 - Homework 10

Due 04/09/09

1. (Problem 2, page 239) Observe that the binomial theorem can be written as follows:

\[(x_1 + x_2)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha \quad (x = (x_1, x_2), \alpha = (\alpha_1, \alpha_2)).\]

Prove the following generalizations:

(a) The multinomial theorem: If \(x \in \mathbb{R}^n\),

\[(x_1 + \cdots + x_n)^k = \sum_{|\alpha| = k} \frac{k!}{\alpha!} x^\alpha.

(b) The \(n\)-dimensional binomial theorem:

\[(x + y)^\alpha = \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} x^\beta y^\gamma.

2. (Problem 3, page 239) Let \(\eta(t) = e^{-1/t}\) for \(t > 0\), \(\eta(t) = 0\) for \(t \leq 0\).

(a) For \(k \in \mathbb{N}\) and \(t > 0\), \(\eta^{(k)}(t) = P_k(1/t)e^{-1/t}\), where \(P_k\) is a polynomial of degree \(2k\).

(b) \(\eta^{(k)}(0)\) exists and equals zero for all \(k \in \mathbb{N}\).

3. (Problem 4, page 239) If \(f \in L^\infty\) and \(\|\tau_y f - f\|_\infty \rightarrow 0\) as \(y \rightarrow 0\), then \(f\) agrees a.e. with a uniformly continuous function. (Let \(A_r f\) be as in Theorem 3.18. Then \(A_r f\) is uniformly continuous for \(r > 0\) and uniformly Cauchy as \(r \rightarrow 0\).)

4. (Problem 7, page 246) If \(f\) is locally integrable on \(\mathbb{R}^n\) and \(g \in C^k\) has compact support, then \(f * g \in C^k\).

5. (Problem 8, page 246) Suppose that \(f \in L^p(\mathbb{R})\). If there exists \(h \in L^p(\mathbb{R})\) such that

\[\lim_{y \rightarrow 0} \|y^{-1}(\tau_y f - f) - h\|_p = 0,\]

we call \(h\) the (strong) \(L^p\) derivative of \(f\). If \(f \in L^p(\mathbb{R}^n)\), \(L^p\) partial derivatives of \(f\) are defined similarly. Suppose that \(p\) and \(q\) are conjugate exponents, \(f \in L^p\), \(g \in L^q\), and the \(L^p\) derivative \(\partial_j f\) exists. Then \(\partial_j (f * g)\) exists (in the ordinary sense) and equals \((\partial_j f) * g\).