Math 5052 - Homework 11

Due 04/16/09

1. (Problem 12, page 254) Work out the analogue of Theorem 8.22 for the Fourier transform on $\mathbb{T}^n$.

2. (Problem 13, page 254) Let $f(x) = \frac{1}{2} - x$ on the interval $[0, 1)$, and extend f to be periodic on $\mathbb{R}$.

   (a) $\widehat{f}(0) = 0$, and $\widehat{f}(\kappa) = (2\pi i\kappa)^{-1}$ if $\kappa \neq 0$.

   (b) $\sum_1^\infty k^{-2} = \pi^2/6$. (Use the Parseval identity.)

3. (Problem 14, page 254 **Wirtinger's Inequality**) If $f \in C^1([a, b])$ and $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |f'(x)|^2 dx.$$

   (By a change of variable it suffices to assume $a = 0, b = \frac{1}{2}$. Extend f to $[-\frac{1}{2}, \frac{1}{2}]$ by setting $f(-x) = -f(x)$, and then extend f to be periodic on $\mathbb{R}$. Check that f, thus extended, is in $C^1(\mathbb{T})$ and apply the Parseval identity.)

4. (Problem 16, page page 255) Let $f_k = \chi_{[-1,1]} * \chi_{[-k,k]}$.

   (a) Compute $f_k(x)$ explicitly and show that $\|f_k\|_u = 2$.

   (b) $f_k^\vee(x) = (\pi x)^{-2} \sin 2\pi kx \sin 2\pi x$, and $\|f_k^\vee\|_1 \to \infty$ as $k \to \infty$. (Use Exercise 15 a, and substitute $y = 2\pi kx$ in the integral defining $\|f_k^\vee\|_1$. Do the calculation of Problem 15 a.)

   (c) $\mathcal{F}(L^1)$ is a proper subset of C_0. (Consider $g_k = f_k^\vee$ and use the open mapping theorem.)

5. (Problem 18, page 255) Suppose $f \in L^2(\mathbb{R})$.

   (a) The L^2 derivative f' (in the sense of Exercises 8 and 9) exists iff $\xi\widehat{f} \in L^2$, in which case $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$.

   (b) If the L^2 derivative f' exists, then

$$\left[\int |f(x)|^2 dx\right]^2 \leq 4 \int |xf(x)|^2 dx \int |f'(x)|^2 dx.$$

   (If the integrals on the right are finite, one can integrate by parts to obtain $\int |f|^2 = -2\mathrm{Re} \int x\overline{f} f'$.)

   (c) (**Heisenberg's inequality**) For any $b, \beta \in \mathbb{R}$,

$$\int (x-b)^2 |f(x)|^2 \, dx \int (\xi - \beta)^2 |\widehat{f}(\xi)|^2 \, d\xi \geq \frac{\|f\|_2^4}{16\pi^2}.$$

   (The inequality is trivial if either integral on the right is infinite; if not, reduce to the case $b = \beta = 0$ by considering $g(x) = e^{-2\pi i\beta x} f(x + b)$.) This inequality, a form of the quantum uncertainty principle, says that f and $\widehat{f}$ cannot both be sharply localized about single points b and β.