Math 5052 - Homework 12

Due 04/23/09

1. (Problem 23, page 256) In this exercise we develop the theory of Hermite functions.

   (a) Define operators T, T^* on $\mathcal{S}(\mathbb{R})$ by $Tf(x) = 2^{-1/2}[xf(x) - f'(x)]$ and $T^*f(x) = 2^{-1/2}[xf(x) + f'(x)]$. Then $\int (Tf)\overline{g} = \int f(\overline{T^*g})$ and $T^*T^k - T^kT^* = kT^{k-1}$.

   (b) Let $h_0(x) = \pi^{-1/4}e^{-x^2/2}$, and for $k \geq 1$ let $h_k = (k!)^{-1/2}T^k h_0$. (h_k is the kth **normalized Hermite function**.) We have $Th_k = \sqrt{k+1}h_{k+1}$ and $T*h_k = \sqrt{k}h_{k-1}$, and hence $TT^*h_k = kh_k$.

   (c) Let $S = 2TT^* + I$. Then $Sf(x) = x^2 f(x) - f''(x)$ and $Sh_k = (2k+1)h_k$. (S is called the **Hermite operator**.)

   (d) $\{h_k\}_0^\infty$ is an orthonormal set in $L^2(\mathbb{R})$. (Check directly that $\|h_0\|_2 = 1$, then observe that for $k > 0$, $\int h_k\overline{h_m} = k^{-1}\int (TT^*h_k)\overline{h_m}$ and use (a) and (b).)

   (e) We have

   $$T^k f(x) = (-1)^k 2^{-k/2} e^{x^2/2}\left(\frac{d}{dx}\right)^k [e^{-x^2/2}f(x)]$$

   (use induction on k), and in particular,

   $$h_k(x) = \frac{(-1)^k}{[\pi^{1/2}2^k k!]^{1/2}} e^{x^2/2}\left(\frac{d}{dx}\right)^k e^{-x^2}.$$

   (f) Let $H_k(x) = e^{x^2/2}h_k(x)$. Then H_k is a polynomial of degree k, called the kth **normalized Hermite polynomial**. The linear span of $H_0, \ldots, H_m$ is the set of polynomials of degree $\leq m$. (The kth Hermite polynomial as usually defined is $[\pi^{1/2}2^k k!]^{1/2}H_k$.)

   (g) $\{h_k\}_0^\infty$ is an orthonormal basis for $L^2(\mathbb{R})$. (Suppose $f \perp h_k$ for all k, and let $g(x) = f(x)e^{-x^2/2}$. Show that $\hat{g} = 0$ by expanding $e^{-2\pi i\xi\cdot x}$ in Maclaurin series and using (f).)

   (h) Define $A : L^2 \to L^2$ by $Af(x) = (2\pi)^{1/4}f(x\sqrt{2\pi})$, and define $\tilde{f} = A^{-1}\mathcal{F}Af$ for $f \in L^2$. Then A is unitary and $\tilde{f}(x) = (2\pi)^{-1/2}\int f(x)e^{-i\xi x}dx$. Moreover $\widetilde{Tf} = -iT\tilde{f}$ for $f \in \mathcal{S}$, and $\tilde{h}_0 = h_0$; hence $\tilde{h}_k = (-i)^k h_k$. Therefore, if $\phi_k = Ah_k$, $\{\phi_k\}_0^\infty$ is an orthonormal basis for L^2 consisting of eigenfunctions for $\mathcal{F}$; namely, $\hat{\phi}_k = (-1)^k\phi_k$.

   [Note: The operator S is the Schrödinger operator, or quantum Hamiltonian, associated to the harmonic oscillator. The eigenvalues of S, up to a physical constant not included above, are the oscillator's quantized energy levels. Hermite functions are also widely used in the theory of stochastic processes.]

2. (Problem 28, page 262) Suppose that $f \in L^1(\mathbb{T})$, and let $A_r f$ be given by (8.38).

   (a) $A_r f = f * P_r$ where $P_r(x) = \sum_{-\infty}^{\infty} r^{|\kappa|}e^{2\pi i\kappa x}$ is the **Poisson kernel** for $\mathbb{T}$.

   (b) $P_r(x) = (1 - r^2)/(1 + r^2 - 2r\cos 2\pi x)$.