Billiards with microstructure: spectrum and diffusion

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Random billiard reflections (definition)

On billiard table with piecewise smooth boundary, we wish to consider **random reflection laws**. Let n be the normal vector to the wall at a collision point p. Define

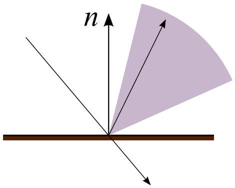
$$\mathcal{H}_p^{\pm} = \{v \text{ tangent vector at } p : \pm\langle v, n\rangle \geq 0\}.$$

A **general random reflection** law is given by a Borel measurable map:

$$v \in \mathcal{H}_p^{-} \mapsto \mu_v \in \mathcal{P}(\mathcal{H}_p^{+})$$

where

$$\mathcal{P}(\mathcal{H}_p^{+}) = \text{ space of Borel probability measures on } \mathcal{H}_p^{+}.$$



Elastic random reflections (definition)

Define

- $\mathcal{S}_p^+$: unit vectors in $\mathcal{H}_p^+$;
- V: volume measure on $\mathcal{S}_p^+$;
- **(Knudsen) cosine law**: $\nu^+ \in \mathcal{P}(\mathcal{S}_p^+)$ such that

$$d\nu^+(u) = c\langle u, n\rangle dV(u).$$

Similarly on $\mathcal{S}_p^-$. We drop the signs for simplicity. An **elastic** random reflection law is an assignment $v \mapsto \mu_v$ such that:

1. **Norm preserving**:

$$v \in \mathcal{S}_p \mapsto \mu_v \in \mathcal{P}(\mathcal{S}_p);$$

2. **Cosine law is invariant**:

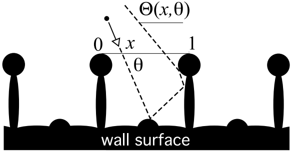
$$\nu = \int_{\mathcal{S}_p} \mu_u \, d\nu(u);$$

3. **Reflection is time-reversible** with respect to ν:

$$d\mu_v(u) \, d\nu(v) = d\mu_{-u}(-v) \, d\nu(-u).$$

Random reflection due to geometric micro-structure

Wall surface has a relief at an **infinitesimal scale**, called a **geometric micro-structure**. For simplicity, let it be periodic. (More realistic geometric models of surfaces can be studied by the same approach.)



Reflection off a surface with micro-structure is defined (dimension 2) by:

$$\mu_\theta(A) = \int_0^1 \chi_A(\Theta(x,\theta))\, dx.$$

Equivalently, Θ is a **random function** of θ given by $\Theta(x,\theta)$, where x is a random variable in $[0,1]$ with the **uniform distribution.** We are identifying $\mathcal{S}$ with $[0,\pi]$. Figure shows ordinary billiard motion inside a **billiard cell** of the micro-structure.

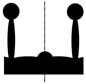
The operator P on functions and measures

Proposition

Reflection off symmetric micro-structure is elastic.

Let P act on functions $f : [0, \pi] \to \mathbb{R}$ by $(Pf)(\theta) = \int_0^1 f(\Theta(x, \theta))\, dx$ and on measures by duality: $(\mu P)(f) = \mu(Pf)$.

- The Knudsen measure ν is **invariant**: $\nu P = \nu$ ($d\nu = \frac{1}{2} \sin\theta\, d\theta$);
- $P : L^2([0, \pi], \nu) \to L^2([0, \pi], \nu)$ has norm $\|P\|_2 = 1$. If the billiard cell is **symmetric**, then P is **self-adjoint**. (Time-reversibility of deterministic billiard.) The spectrum of P is contained in $[-1, 1]$.



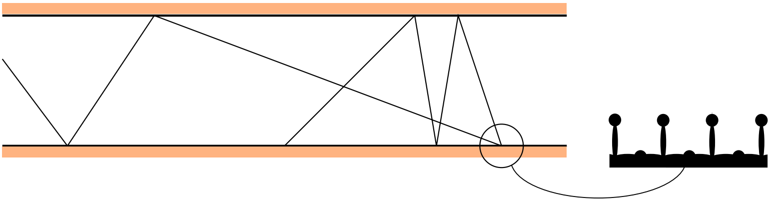
- P is fully specified by the shape of the billiard-cell. Thus we can speak of the **spectrum of the billiard micro-structure**.

Random flights in a channel

We regard P as the transition probabilities operator for Markov chains

$$\Theta_1, \Theta_2, \Theta_3, \ldots$$

with state space $[0, \pi]$, describing the collision angles of a random billiard trajectory in a channel.



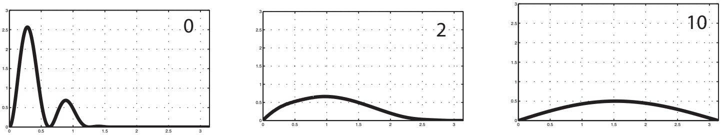
For long and narrow channels, random flight should limit to **Brownian motion** on the real line. (More on C.L.T. later.)

Convergence to stationary distribution ($d\nu = \frac{1}{2} \sin \theta \, d\theta$)

Proposition

Under very general conditions (e.g., if the most exposed point has positive curvature), then the Markov chain is irreducible and aperiodic and for any initial angle distribution ν_0 we have convergence $\nu_0 P^n \to \nu$.

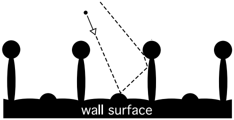
Stationary probability is the same for all shapes of the billiard cell.



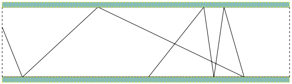
(0) arbitrary initial density; (2) after 2 collisions; (10) after 10 collisions for P associated to the shape:

Gas transport in channels - 3 levels of description

- Hard-ball model at micro level (Deterministic billiard motion)



- Random flight in channel (Markov process on set of directions)



- Diffusion limit: gas concentration $u(x, t)$ along $\mathbb{R}$ should satisfy

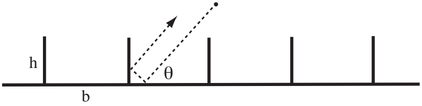
$$\frac{\partial u}{\partial t} = \mathcal{D} \frac{\partial^2 u}{\partial x^2}$$

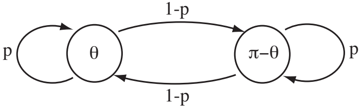
Relate: (1) **shape of cell**, (2) **spectrum of** P, (3) **diffusion constant** $\mathcal{D}$.

An elementary example: the entire program in a nutshell

Let θ be the initial angle and define integer k, $s \in [0, 1)$, and probability p:

$$\frac{2h}{b\tan\theta} = k + s, \qquad p = \begin{cases} s & \text{if } k \text{ is odd} \\ 1 - s & \text{if } k \text{ is even} \end{cases}$$





Special case: $\theta = \pi/4$, $b > 2h$. Then $k = 0, s = 2h/b$. For a long channel of diameter $2r$ and particle speed v, the random flight tends to Brownian motion with

$$\sigma^2 = \sqrt{2}rv\left(\frac{b}{2h} - 1\right).$$

This is an application of the **central limit theorem** for Markov chains.

Transition to diffusion—the Central Limit Theorem

Consider the following experiment. Let

- r = radius of channel;
- v = constant particle speed;
- L = half channel length;

Release the particle from middle point with distribution ν. Measure the **expected exit time**, $\tau(aL, r, v)$ as $a \to \infty$.

Proposition

Suppose P on $L^2([0,\pi],\nu)$ has positive spectral gap and ν is ergodic for P. Then

$$\tau(aL, r, v) \sim \frac{1}{\mathcal{D}} \frac{a^2}{\ln a}$$

where $\mathcal{D} = \frac{4rv}{\pi}\xi(P)$.

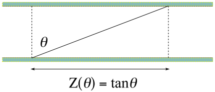
We wish to understand how $\mathcal{D}$ depends on P.

$\mathcal{D}$ and the spectrum of P

Let Π be the (projection-valued) spectral measure of P on $[-1,1]$:

$$P = \int_{-1}^{1} \lambda \, d\Pi(\lambda).$$

Fix $\beta > 1$ and let $Z|_a = Z\chi_{\{|Z| \leq a/\ln^\beta a\}}$.



Spectral measure of Z on $[-1,1]$: $\Pi_Z(\cdot) = \lim_{a\to\infty} \frac{1}{\ln a}\langle Z|_a, \Pi(\cdot) Z|_a\rangle$.

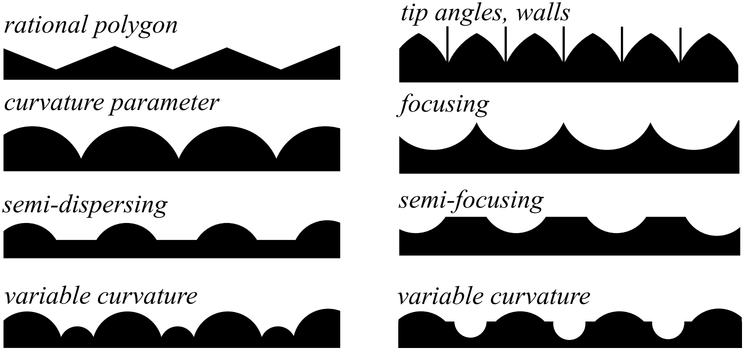
Theorem

Let $\mathcal{D}_0 =$ *diffusion const. for i.i.d. process with angle distribution* ν. *Then*

$$\mathcal{D} = \mathcal{D}_0 \int_{-1}^{1} \frac{1+\lambda}{1-\lambda} \, d\Pi_Z(\lambda).$$

If P *has* **discrete spectrum**, $\Pi_Z(\lambda_i) := \lim_{a\to\infty} \frac{1}{\ln a} |\langle Z|_a, \phi_i\rangle|^2$.

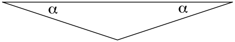
Plan: find the effect on spectrum of various shape features



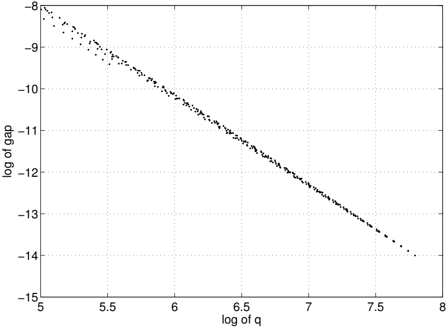
Relation between shape and spectrum is, for the moment, a case-by-case study, but there are some general ideas.

Rational triangles (only numerical)

Billiard cell is an isosceles triangle with **rational angle** $\alpha = \frac{p\pi}{q}$.



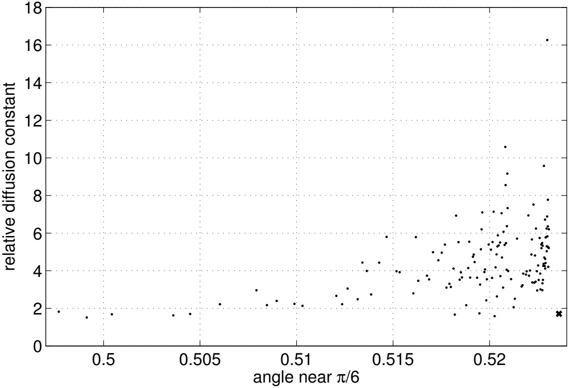
Numerical observation: $|\text{gap}| \sim 12.2q^{-2.1}$.



$\log\gamma$ as a function of α for $\alpha/\pi = [0; m_1, m_2, m_3]$, $m_1 = 6$, $m_2, m_3 \le 20$.

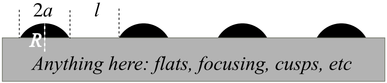
Diffusion constant $\mathcal{D}/\mathcal{D}_0$ for rational triangles

Values of the relative diffusion constant $\mathcal{D}/\mathcal{D}_0$ for triangle with angles $\alpha = \frac{\lfloor q/6 \rfloor}{q}\pi$, where q runs over the set of primes between 100 and 1000. So these are all approximations of $\alpha = \pi/6$.



General spectral properties of P: Bumps family

Bumps family: symmetric micro-relief with $0 \leq l$, $0 < a \leq R$, $2a + l = L$.



Simple bumps family: $l = 0$.

Theorem

For the bumps family of billiard cells,

- *the spectrum of P lies in* $[-1, 1]$*;*
- 1 *is a* **simple eigenvalue**, -1 *not an eigenvalue;*
- *eigenfunctions for simple eigenvalues are either* **odd or even***:*

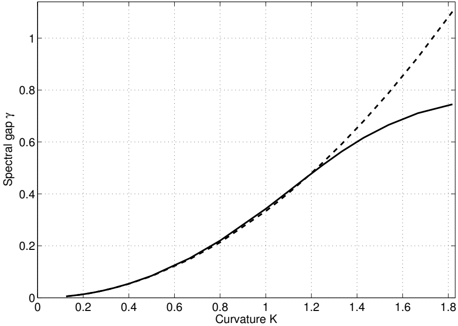
$$\varphi(\theta) = \pm\varphi(\pi - \theta);$$

- P *is* **quasi-compact** *(compact if $l = 0$), hence* **positive spectral gap***;*
- *if $a = R$ then* **gap** $\geq 1/7K$ *where $K = (l + 2a)/R$.*

The simple bumps family for small K: spectral gap

Theorem

Spectral gap for the simple bumps family: $\gamma(K) = \frac{1}{3}K^2 + \mathcal{O}(K^3)$.



Comparison of $\gamma(K)$ (solid line) and $\frac{1}{3}K^2$ (dashed line).

Small K—a perturbation approach to the spectrum

Define the j**th moment of scattering**:

$$\mathcal{E}_j(\theta) = E(\Theta - \theta)^j,$$

where Θ is the random **post-collision angle** and θ the **pre-collision** angle. Small values of K correspond to small values of $\mathcal{E}_j$.

These moments determine the operator P. If $\Phi : [0, \pi] \to \mathbb{R}$ is a smooth function, the **billiard Laplacian** $P - I$ acting on Φ can be approximated by

$$P\Phi - \Phi = \sum_{j=1}^{n} \frac{\Phi^{(n)}}{n!} \mathcal{E}_j + \mathcal{O}(\mathcal{E}_{n+1}).$$

Moment estimates for the simple bumps family

Theorem

If $\sin\theta > 3K/2$ *(middle range of angles), the moments satisfy:*

- *If n is odd,*

$$\mathcal{E}_n(\theta) = \frac{K^{n+1}}{2(n+2)}\cot\theta + \mathcal{O}(K^{n+2});$$

- *If n is even,*

$$\mathcal{E}_n(\theta) = \frac{K^n}{n+1} + \mathcal{O}(K^{n+1}).$$

Also obtain bounds on moments for $\sin\theta \leq 3K/2$.

Approximation of the billiard Laplacian $P - I$

Theorem

If Φ is smooth, compactly supported in $(0, \pi)$, then

$$P_K\Phi - \Phi = \mathcal{E}_1\Phi' + \frac{1}{2}\mathcal{E}_2\Phi'' + \mathcal{O}(\mathcal{E}_3) = \frac{K^2}{6}\mathcal{L}_0\Phi + \mathcal{O}(K^3),$$

where

$$\mathcal{L}_0\Phi = \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Phi}{d\theta}\right).$$

The **diffusion process on the interval** $(0, \pi)$ associated to $\mathcal{L}_0$ is

$$d\Theta_t = \frac{1}{2}\cot\Theta_t dt + dB_t, \quad P^{\lfloor t/\gamma_k \rfloor} \sim e^{t\mathcal{L}_0}$$

where B_t is standard Brownian motion and

$$\text{drift} \to \begin{cases} +\infty & \text{for } \theta \to 0 \\ -\infty & \text{for } \theta \to \pi \end{cases}$$

Interpretation: billiards and spherical harmonics

The (reduced) billiard map is naturally viewed as an area-preserving map

$$T : S^2 \to S^2.$$

In fact, if $(s, \theta) \in [0,1] \times [0, \pi]$ are billiard coordinates, then $(\theta, \phi = 2\pi s)$ are spherical coordinates on S^2 where ϕ is longitude and θ is latitude (measured from the North pole). Then the normalized area measure

$$\frac{dA}{4\pi} = \frac{1}{2} \sin \theta d\theta ds$$

is the billiard Liouville measure. The Markov operator P can be regarded as a self-adjoint operator on $L^2(S^2, A)$.

Theorem

Let Φ be a compactly supported smooth function on $S^2 \setminus \{N, S\}$ invariant under rotations about the z-axis in $\mathbb{R}^3$. Then

$$P_K\Phi - \Phi = \frac{K^2}{6}\Delta\Phi + \mathcal{O}(K^3).$$

Billiard eigenfunctions, Legendre polynomials and gap

The equation

$$\mathcal{L}_0\Phi + l(l+1)\Phi = 0$$

is the **Legendre equation** (for $m = 0$) and its canonical solutions for integer $l \geq 0$ are the **Legendre functions** Φ_l:

$$\Phi_0(\theta) = 1,\ \Phi_1(\theta) = \cos\theta, \Phi_2(\theta) = \frac{1}{2}(3\cos^2\theta - 1), \ldots$$

These are approximate (right) eigenfunctions of the billiard Laplacian:

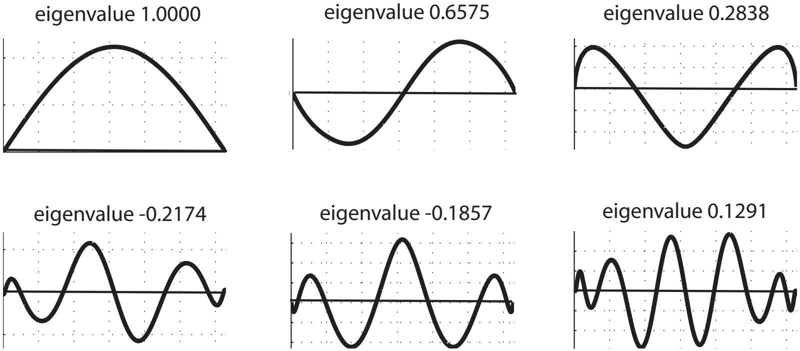
$$P_K\Phi_l \sim \left(1 - \frac{K^2}{6}\ l(l+1)\right)\Phi_l.$$

The second eigenvalue is

$$\lambda_2 \sim 1 - \frac{K^2}{3}\ \ (l = 1)$$

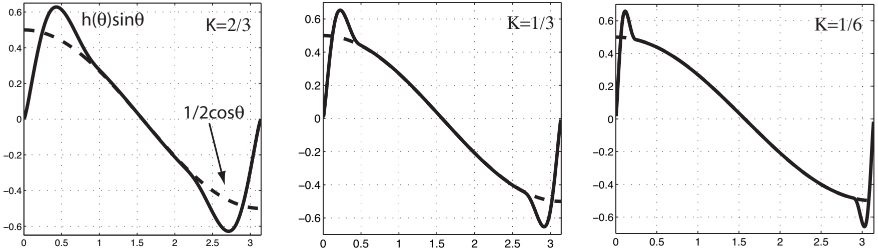
and the associated eigenfunction is $\Phi_1(\theta) \sim \cos\theta$ and spec. gap $\sim \frac{K^2}{3}$.

Top eigenfunctions for simple bumps, $K = 1$



Let $h(\theta) = \mathcal{E}_1/\gamma_K$. The gaps for $K = 2/3, 1/3, 1/6$ are

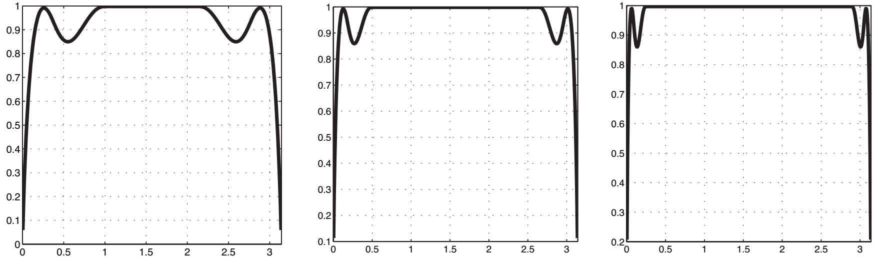
$$\gamma_K = 0.1519,\ 0.0373,\ 0.0093.$$



Comparing $h(\theta)\sin\theta$, $\frac{1}{2}\cos\theta$ for $K = 2/3$, $K = 1/3$, $K = 1/6$.

$\mathcal{E}_2(\theta)/\gamma_K$ for $K = 2/3, 1/3, 1/6$

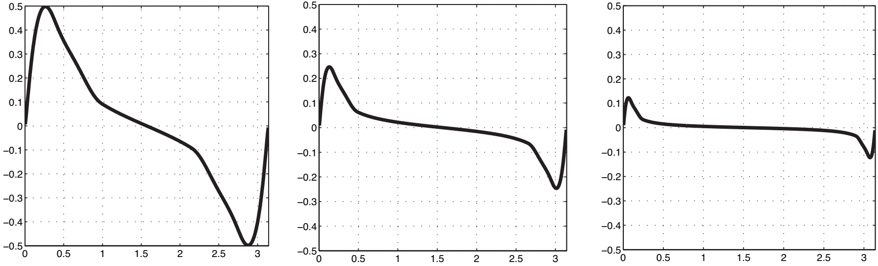
$E(\Theta_K - \theta)^2/(\text{spectral gap}) \rightarrow$ constant function 1:



This gives an asymptotic interpretation of the spectral gap as the mean square dispersion of the post-collision angle about the pre-collision angle.

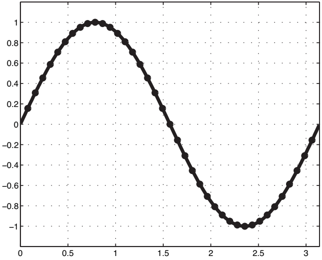
Error term $\mathcal{E}_3(\theta)/\gamma_K$ for $K = 2/3, 1/3, 1/6$

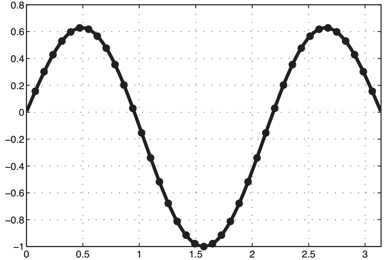
The error term goes to 0 as $K \to 0$.



Comparison of eigenfunctions

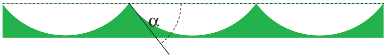
Comparison of **second and third left-eigenfunctions** of P_K (solid) and $\Phi_j(\theta)\sin\theta$ (circles), for $j = 1, 2$ and $K = 1/6$.



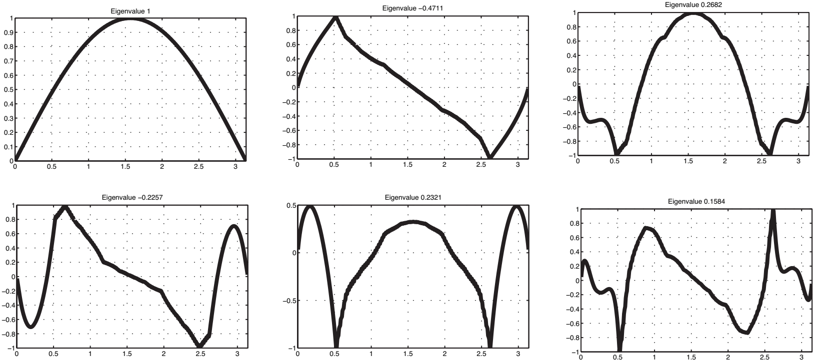


Equality holds for the eigenfunctions with eigenvalue $\lambda = 1$. Top four numerically obtained eigenvalues agree to 4 decimals with corresponding quantities obtained using the Legendre equation.

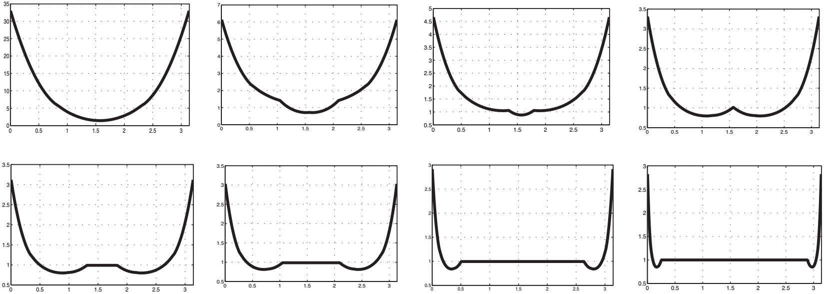
Case study II: focusing billiard



Some of the top eigenvalues and respective eigenfunctions for $\alpha = 5\pi/12$.



$\mathcal{E}_2(\theta)/\gamma_K$ for the family of focusing billiards



K varies from 2.0 for top left graph to 0.25 for lower right graph.