Rigid Geometric Structures
and Actions of Semisimple Lie Groups

R. FERES[*]

July 19, 2000

1   Introduction

Let M be a smooth manifold, $\mathcal{G}$ a geometric structure on M, and G a Lie group that acts on M so as to leave $\mathcal{G}$ invariant. In other words, G is a group of isometries of $(M, \mathcal{G})$. The subject of these lectures will be the general interplay between the dynamics of the G-action on M and the geometry of $(M, \mathcal{G})$, as well as how these relate to the topology of M. The case of a semisimple G will be of particular interest to us.

Even with the constraint that G be semisimple, the objective stated above will certainly seem too general and vague. There are, however, some interesting and effective ideas that apply to a large class of geometric structures and actions. The main results that will be discussed here are due to M. Gromov, and are introduced and further explored in his paper [10]. The techniques that relate more directly to the dynamics of semisimple Lie groups, in the way that is used here, are due to R. Zimmer. Put together, their ideas form somewhat of a coherent framework, and it is the main point of these lectures to give an introduction to this Gromov-Zimmer machine, and to illustrate its use with a small number of representative results.

This is not meant to be a survey of the area. The main theorem of the notes is Gromov's theorem 4.1, although some of the ingredients used in its proof will be of independent interest. Many more related results can be found by going to the main sources of the lectures: [23], [10], [5], [22], and the references cited in them.

---

[*]Department of Mathematics, Washington University, St. Louis, MO 63130 - USA. (feres@@math.wustl.edu).

[*]1

The first section (section 2) deals with the basic language and facts about geometric structures and their isometries. Sections 3 and 4 introduce the notion of a rigid structure and states a fundamental result of Gromov's relating infinitesimal and local isometries, whose proof is provided in detail in section 5. The relationship between dynamics and geometry is the main theme of section 6. The results given there are mostly due to Zimmer. Finally, section 7 brings the fundamental group of the manifold into the picture by showing that nontrivial actions of semisimple Lie groups by isometries of rigid unimodular analytic structures can only be supported by manifolds with "large" fundamental groups.

Some background material concerning dynamical systems, algebraic group actions, and semisimple Lie groups is provided in the appendices. The text [6] can also serve as a useful introduction to part of the subject of the present notes.

2   Geometric structures

We introduce here a general definition of geometric structure on manifolds and illustrate it with a number of examples. The material presented here is, for the most part, either standard (see, for example, [14]) or comes from [10], or both. It will be assumed that the reader has some familiarity with the language of principal bundles and jets, although no deep facts concerning them will be used here. A good reference for that subject is [16]. We introduce in the next paragraph some notation and first definitions on jets.

Let M and N be smooth manifolds and let f, g be smooth maps from some neighborhood of a point $x \in M$ into N. We say that f and g represent the same r-jet at x if $f(x) = g(x) = y$ and with respect to some choice of smooth coordinates near x and y, all partial derivatives of f and g up to order r agree. More precisely, let t^i, $1 \leq i \leq \dim M$, be smooth coordinates near x, let u^j, $1 \leq j \leq \dim N$, be smooth coordinates near y, and represent by D_i the partial derivative with respect to t_i. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector of nonnegative integers and define $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. Then f and g represent the same r-jet at x if for each i and α such that $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq r$

$$D^\alpha(u^i \circ f)_x = D^\alpha(u^i \circ g)_x.$$

This defines an equivalence relation on C^r local maps, which we denote by

$$f \underset{x}{\overset{r}{\sim}} g.$$

The equivalence relation does not depend on the choice of coordinates. The equivalence class represented by f is called the r-jet of f at x and is denoted $j^r f_x$. The r-jets of local C^r maps comprise the r-jet space $J^r(M, N)$. This is a smooth manifold. Smooth local coordinates can be set on $J^r(M, N)$ as follows:

$$u^i_\alpha(j^r f_x) := D^\alpha(u^i \circ f)_x.$$

The r-jet of a smooth vector field is also defined, by regarding the vector field as a map (section) from M into TM.

A (smooth) parametrization of an open subset $U \subset M$ is a (smooth) diffeomorphism from an open subset of $\mathbb{R}^n$ onto U. We say that $\varphi : U_0 \to U$ is a parametrization at x if $0 \in U_0$, $x \in U$ and $\varphi(0) = x$.

A *frame of order r* at $x \in M$ is the r-jet at x of a smooth parametrization at x. A frame of order 1 at x is naturally identified with a linear isomorphism from $\mathbb{R}^n$ onto the tangent space $T_x M$. In the general case, the equivalence class represented by a parametrization φ will be denoted $(j^r \varphi)_0$ – the r-th jet of φ at 0.

The collection of all frames over points of M forms in a natural way a smooth manifold, which will be called the r-th order *frame bundle of M* and will be denoted $F^r(M)$. This is indeed a locally trivial fiber bundle over M and the bundle map $\pi : F^r(M) \to M$ is the obvious base point projection, which to each $(j^r \varphi)_0$ associates $\varphi(0)$.

Having fixed a frame $\xi = (j^r \varphi)_0$ at x, any other frame of order r at the same point is given by ξg, where $g = (j^r f)_0$ is the r-jet of a diffeomorphism f from a neighborhood of 0 into another neighborhood of 0 such that $f(0) = 0$. By definition,

$$\xi g := j^r(\varphi \circ f)_0.$$

The collection of all r-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ fixing 0 forms a Lie group, denoted here $G^r = G^r(n, \mathbb{R})$. Notice that G^1 is the general linear group $GL(n, \mathbb{R})$. It can be shown that G^r is in a natural way a linear real algebraic group. (G^r can be regarded as a subgroup of $GL(V^{r-1})$ of all invertible linear transformation of the vector space of $r - 1$ jets at 0 of smooth vector fields on $\mathbb{R}^n$.)

The map $F^r(M) \times G^r \to F^r(M)$ given by $(\xi, g) \mapsto \xi g$ is a smooth group action that sends each fiber of $F^r(M)$ onto itself. It is clear, furthermore, that the action is transitive on each fiber. With this action, $F^r(M)$ becomes a principal bundle. A smooth parametrization of an open subset $U \subset M$ can be used to trivialize $F^r(M)$ above U, making $\pi^{-1}(U) \subset F^r(M)$ isomorphic to the trivial bundle $U \times G^r$.

We are now ready to give the definition of a geometric structure on M. Let V be a space (with further structure to be specified later) equipped with a (left) action of G^r. A *geometric structure on M of order r and type V* is a map $\mathcal{G} : F^r(M) \to V$ that satisfies the G^r-equivariance property:

$$\mathcal{G}(\xi g) = g^{-1}\mathcal{G}(\xi).$$

When V is a smooth manifold, we say that the geometric structure is smooth (resp. C^r, real analytic, continuous, measurable, etc.) if the equivariant map $\mathcal{G}$ is smooth (resp. C^r, real analytic, continuous, measurable).

A geometric structure (of any degree of regularity) is an *A-structure* or a *structure of algebraic type* if V is a smooth real algebraic variety and the left action of G^r on V is also algebraic. As will be seen in the examples below, geometric structures ordinarily considered in differential geometry are A-structures.

We give next a collection of examples. In each case, M is a manifold of dimension n and $F(M) = F^1(M)$.

0. Structures of order 0. By convention, $F^0(M) = M$ and G^0 is the trivial group. Then, a structure of order 0 and type V is simply a map $\mathcal{G} : M \to V$.

1. L-structures. Let L be a closed subgroup of G^r and $P \subset F^r(M)$ a subset such that the restriction of π^r to P is a principal L-bundle. Each $\xi \in F^r(M)$ is naturally associated to a coset in G^r/L so that P defines a G^r-equivariant map $\mathcal{G} : F^r(M) \to V$, where $V = G^r/L$. We say that $\mathcal{G}$ is an *L-structure of order r*. Conversely, if $\mathcal{G} : F^r(M) \to G^r/L$ is an equivariant map, it is easily seen to produce an L-reduction of $F^r(M)$, i.e., a principal subbundle $P \subset F^r(M)$ with group L such that the right L-action on P is the restriction of the G^r-action on $F^r(M)$. In fact, $P = \mathcal{G}^{-1}(L)$, where L represents here the identity coset in G^r/L. The reader will notice that most of the examples given next are L-structures.

2. Complete parallelism. This is defined by the assignment of a linear frame (linear isomorphism) $\sigma(x) : \mathbb{R}^n \to T_x M$ to each $x \in M$ – in other words: a section of $F(M)$. Alternatively, complete parallelism can be defined by a $GL(n, \mathbb{R})$-equivariant map $\mathcal{G} : F(M) \to GL(n, \mathbb{R})$, where $GL(n, \mathbb{R})$ acts on $V = GL(n, \mathbb{R})$ by left-multiplication. The relationship between $\mathcal{G}$ and σ is that

$$\xi = \sigma(x)A \Leftrightarrow \mathcal{G}(\xi) = A^{-1}.$$

A complete parallelism on M is an L-structure of order 1, where L is the trivial group.

More generally, a complete parallelism of $F^r(M)$ defines a geometric structure of order $r + 1$. Here $V = GL(N, \mathbb{R})$, where N is the dimension of $F^r(M)$ and the map $\mathcal{G} : F^{r+1}(M) \to V$ is defined as follows. For each $\xi \in F^{r+1}(M)$, set $\bar{\xi} = \pi_r^{r+1}(\xi) \in F^r(M)$; notice that ξ gives rise to a linear subspace of $T_{\bar{\xi}} F^r(M)$ transverse to the fiber of $F^r(M) \to M$ at $\bar{\xi}$, which we denote $L(\xi)$. The linear isomorphism $d\pi_{\bar{\xi}}^r : L(\xi) \to T_x M$ defines on $L(\xi)$ a frame given by $d\pi_{\bar{\xi}}^r \circ \pi_1^{r+1}(\xi)$. On the other hand, the tangent space of the fiber of $F^r(M)$ at $\bar{\xi}$ is naturally isomorphic to the Lie algebra of G^r, due to the principal action of G^r on $F^r(M)$. Therefore, we obtain in a canonical way a frame of $T_{\pi_r^{r+1}(\xi)} F^r(M)$. The desired $\mathcal{G}(\xi)$ is now the change of basis matrix between the two frames: the one just constructed and the one given by a parallelism of $F^r(M)$.

3. Volume-form. By a volume form we mean a (smooth, C^r, continuous, measurable, etc.) assignment of a non-vanishing n-form on each $T_x M$. It can be given by an equivariant map $\mathcal{G} : F(M) \to V$ where V denotes the space of non-zero alternating n-forms on $\mathbb{R}^n$. Let μ_0 be the n-form

$$\mu_0(u_1, \ldots, u_n) = \det(u_{ij})$$

where u_{ij} are the entries of the matrix whose columns are the vectors $u_1, \ldots, u_n \in \mathbb{R}^n$. The general linear group acts transitively on V and the isotropy subgroup of μ_0 is $SL(n, \mathbb{R})$. An $SL(n, \mathbb{R})$-reduction of $F(M)$ determines a volume form on M, and the equivariant map $\mathcal{G}$ defined by ν is given as follows: to each $\xi \in F(M)_x$, $\mathcal{G}(\xi)$ is the n-form on $\mathbb{R}^n$ given by $\nu(x)(\xi\cdot, \ldots, \xi\cdot)$.

4. Pseudo-Riemannian metric. Here, V is the space of nondegenerate symmetric bilinear forms on $\mathbb{R}^n$ of signature s. The (transitive) action of $GL(n, \mathbb{R})$ corresponds to left-multiplication on $GL(n, \mathbb{R})/O(p, n-p)$, where $s = 2p - n$ is the signature and $O(p, n-p)$ is the isotropy subgroup of

$$\beta_0(u, u) := u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots - u_n^2.$$

(We are identifying here a transitive G-space and the corresponding homogeneous space.) The action of $GL(n, \mathbb{R})$ on V can be written as

$$gB = B(g^{-1}\cdot, g^{-1}\cdot).$$

The map $\mathcal{G} : F(M) \to V$ corresponds to the assignment of a symmetric bilinear form $\langle \cdot, \cdot \rangle_x$ on each tangent space $T_x M$ by the relation

$$\langle v, w \rangle_x = \mathcal{G}(\xi)(\xi^{-1}v, \xi^{-1}w)$$

5

where ξ is any element in the fiber above x. $GL(n,\mathbb{R})$-equivariance of $\mathcal{G}$, which translates to the property $\mathcal{G}(\xi g)(\cdot,\cdot) = \mathcal{G}(\xi)(g\cdot, g\cdot)$, implies that $\langle\cdot,\cdot\rangle_x$ does not depend on the choice of ξ.

If instead of $O(p, n-p)$ we use the group $CO(p, n-p)$ that leaves β_0 invariant up to scalar, we obtain the definition of a *conformal pseudo-Riemannian* structure.

5. Tensor fields. Let ρ be a linear representation of $GL(n,\mathbb{R})$ on W, $\rho : GL(n,\mathbb{R}) \to GL(W)$. A tensor field of type W may be defined by a $GL(n,\mathbb{R})$-equivariant map $\mathcal{G} : F(M) \to W$, where equivariance in this case is defined by

$$\mathcal{G}(\xi A) = \rho(A)^{-1}\mathcal{G}(\xi)$$

for all $A \in GL(n,\mathbb{R})$ and $\xi \in F(M)$. For example, when

$$W = (\overset{r}{\bigotimes}\mathbb{R}^{n*}) \otimes (\overset{s}{\bigotimes}\mathbb{R}^n)$$

and ρ is the representation obtained from the standard representation of $GL(n,\mathbb{R})$ on $\mathbb{R}^n$, the map $\mathcal{G}$ is equivalent to having a section of the vector bundle $T^{(r,s)}M$ of tensors of type (r,s). A section of $T^{(r,s)}M$ is called a tensor field of type (r,s). Notice that a vector field is a section of $T^{(0,1)}M$ and an n-form is a section of $T^{(n,0)}M$.

Tensor fields are not necessarily L-structures. For example, a field of endomorphisms, that is, a tensor field of type $(1,1)$, is an L-structure exactly when it has the same Jordan normal form, $J \in GL(n,\mathbb{R})$, at every point of M, in which case L is the centralizer of J in $GL(n,\mathbb{R})$.

6. Subbundles of TM. Viewing $\mathbb{R}^m$ as the subspace of $\mathbb{R}^n$ of vectors with the last $n-m$ components equal to 0, we let $GL(n,m,\mathbb{R})$ denote the subgroup of $GL(n,\mathbb{R})$ consisting of invertible matrices that map $\mathbb{R}^m$ into itself. The group $GL(n,m,\mathbb{R})$ is the isotropy subgroup of $\mathbb{R}^m$ for the transitive action of $GL(n,\mathbb{R})$ on the Grassmannian variety V of m-dimensional subspaces of $\mathbb{R}^n$. A smooth $GL(n,m,\mathbb{R})$-reduction of $F(M)$ corresponds to a smooth field $x \mapsto D(x)$ of m-dimensional subspaces $D(x)$ of T_xM. The equivariant map $\mathcal{G}$ associated to D sends each $\xi \in F(M)$ to the subspace $\xi^{-1}D(x) \subset \mathbb{R}^n$, $x = p(\xi)$.

Before considering some examples of second order structures, we need to understand a little better the group G^2. The second order jet of a local diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ fixing the origin of $\mathbb{R}^n$ is completely specified by a pair (A, α) such that $A \in GL(n,\mathbb{R})$ and $\alpha \in N^2 := S^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ is an $\mathbb{R}^n$-valued symmetric bilinear form. Indeed, we can think of (A, α) as the Taylor polynomial of φ of order 2. (More precisely, the second degree $\mathbb{R}^n$-valued polynomial approximating φ is $Ax + \alpha(x, x)$.) If φ and ψ have Taylor

6

polynomials (A, α) and (B, β), respectively, then the composition $\varphi \circ \psi$ has Taylor polynomial $(AB, \alpha(B\cdot, B\cdot) + \beta)$. The group G^2 can, therefore, be given the following description. $G^2 = GL(n, \mathbb{R}) \ltimes N^2$, with multiplication given by:

$$(A, \alpha)(B, \beta) := (AB, \alpha(B\cdot, B\cdot) + A\beta).$$

It follows that the inverse operation is $(A, \alpha)^{-1} := (A^{-1}, -A^{-1}\alpha(A^{-1}\cdot, A^{-1}\cdot))$.

In the general case, G^r can be described inductively as a semidirect product of G^{r-1} with the vector space (abelian group) $S^r(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ of symmetric n-linear forms on $\mathbb{R}^n$ taking values in $\mathbb{R}^n$.

7. *Linear connections.* Let V be the space of all $\mathbb{R}^n$-valued bilinear maps $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, which we regard as the space of Christoffel symbols. To describe the action of G^2 on V first notice that N^2 is a linear subspace of V and that G^2 is a closed subgroup of $GL(n, \mathbb{R}) \ltimes V$, where the later has the same multiplication and inverse operations as for G^2. Then V can be identified with the coset space

$$W = GL(n, \mathbb{R}) \backslash GL(n, \mathbb{R}) \ltimes V$$

by setting $\Gamma \mapsto GL(n, \mathbb{R})(I, \Gamma)$. The group G^2 naturally acts on W by right multiplication, yielding the following (right) action on V:

$$\Gamma \cdot (A, \alpha) := A^{-1}\Gamma(A\cdot, A\cdot) + A^{-1}\alpha.$$

(We can easily rewrite the above so as to turn it into a left action, by setting $g \cdot \Gamma = \Gamma g^{-1}$.) Notice that this is the familiar law of transformation of Christoffel symbols. Thus, in the language used in these notes, a linear connection corresponds to a G^2-equivariant map $\mathcal{G} : F^2(M) \to V$. The relationship between $\mathcal{G}$ and the notion of a covariant derivative is that if $\xi \in F^2(M)$ is the 2-jet (at 0) of a smooth parametrization φ around $x \in M$ (with $\varphi(0) = x$) and X_i are the coordinate vector fields associated to φ, then

$$(\nabla_{X_i} X_j)_x = \sum_{k=1}^{n} \mathcal{G}(\xi)_{ij}^k X_k$$

where $\mathcal{G}(\xi)_{ij}^k$ is the k-th component of $\mathcal{G}(\xi)(e_i, e_j)$ with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. Also notice that a symmetric (torsion-free) connection is one for which the map $\mathcal{G}$ takes values into $GL(n, \mathbb{R}) \backslash G^2 \subset V$.

It is not difficult to show that a linear connection gives rise to a subbundle of $TF^1(M)$ everywhere transverse to the fibers (a horizontal distribution), and therefore, to a complete parallelism of $F^1(M)$. One may define a *generalized connection* of order r on M as a horizontal distribution on $F^r(M)$.

7

This is a structure of order $r + 1$ that gives rise to a complete parallelism on $F^r(M)$.

8. Projective structures. A (smooth) projective structure is an equivalence class of (smooth) connections, two connections being equivalent if and only if they define the same geodesic lines, without regard to parameter. Since we can also find a torsion-free connection in each such equivalence class, we may assume that the equivalence relation is on the space of symmetric connections. A projective structure may also be defined by an equivariant map $\mathcal{G} : F^2(M) \to V$, where V is the coset space $H \backslash G^2$, with the natural right action of G^2, and H is the subgroup of G^2 of all (A, α) such that

$$\alpha = -\frac{A \otimes \sigma + \sigma \otimes A}{2}$$

for a linear functional σ on $\mathbb{R}^n$.

9. Symplectic structure. A symplectic form is a non-degenerate 2-form with the added assumption that it is closed. A smooth alternating and nondegenerate 2-form is clearly a first order structure and is a special case of example 5. The condition that it is a closed form can be incorporated by describing it as a second order structure, as follows. Let Λ be the space of alternating bilinear forms on $\mathbb{R}^n$. The linear space $\Lambda \oplus [\Lambda \otimes (\mathbb{R}^n)^*]$ may be regarded as the space of 1-jets (at 0) of 2-forms in $\mathbb{R}^n$. Given (ω, ν) in $\Lambda \oplus [\Lambda \otimes (\mathbb{R}^n)^*]$, and vectors $u_1, u_2, u_3 \in \mathbb{R}^n$, then $\nu(u_1, u_2, u_3) =: \nu(u_1, u_2)u_3$ is regarded as the directional derivative at 0 of the smooth 2-form evaluated on the constant vector fields u_1, u_2 along the direction determined by u_3. The 1-jet of a closed 2-form is given by a pair (ω, ν) such that for all u_1, u_2, u_3 in $\mathbb{R}^n$ the relation

$$\nu(u_1, u_2)u_3 + \nu(u_2, u_3)u_1 + \nu(u_3, u_1)u_2 = 0$$

holds. We now define V as the subset of $\Lambda \oplus [\Lambda \otimes (\mathbb{R}^n)^*]$ of all pairs (ω, ν) such that ω is nondegenerate and ν satisfies the above identity. The group G^2 acts on V as follows (this corresponds to the pull-back of forms): $(\omega, \nu)(A, \alpha) = (\omega', \nu')$ where $\omega_1 = \omega(A\cdot, A\cdot)$ and for $u_1, u_2, u_3 \in \mathbb{R}^n$,

$$\nu'(u_1, u_2)u_3 = \nu(Au_1, Au_2)u_3 + 2[\omega(Au_1, \alpha(u_2, u_3)) + \omega(\alpha(u_1, u_3), Au_2)].$$

A symplectic form is now a G^2-equivariant map $\mathcal{G} : F^2(M) \to V$.

10. Homogeneous structures. Suppose that $M_0 = G/H$, where G is a connected Lie group and H is a connected closed subgroup of G. We suppose that H does not contain a proper normal subgroup of G, which implies that the action of G on M_0 by left multiplication is an effective transitive action.

(An action is said to be effective if for each $g \in G$ there exists $x \in M_0$ such that $gx \neq x$.) It can be shown that there exists a positive integer k such that the natural action of G on $F^k(M_0)$ is free and proper, so that we can define the quotient $V = G \backslash F^k(M_0)$. Notice that V admits a right action of G^k coming from the action on the $F^k(M_0)$. A homogeneous structure of order k and type (G, M_0) on a manifold M is defined as a G^k-equivariant map $\mathcal{G} : F^k(M) \to V$. This corresponds to the assignment, for each $x \in M$, of an equivalence class of k-jets at x of local diffeomorphisms from a neighborhood of x into M_0 under the equivalence relation defined by the natural action of G on the space of such jets.

11. Itô vector fields. An interesting example of a second order geometric structure arises in the study of stochastic processes on manifolds. Define on $V = \mathbb{R}^n \times GL(n, \mathbb{R})$ the G^2-action:

$$(A, \alpha)(u, L) = (Au + \mathrm{Tr}\,\alpha(L\cdot, L\cdot)/2, AL)$$

for $(A, \alpha) \in G^2$ and $(u, L) \in V$, where $\mathrm{Tr}\,\alpha(L\cdot, L\cdot) = \sum_{i=1}^n \alpha(Le_i, Le_i)$ for an orthonormal basis of $\mathbb{R}^n$. A G^2-equivariant map $\mathcal{G} : F^2(M) \to V$ specifies an (Itô) stochastic differential equation on M. (See [7] and the references cited there.)

We give next a couple of examples of non-A structures. Perhaps surprisingly, example 13 behaves in some important respects like an A-structure. (See [22], chapter 3.)

12. Anosov diffeomorphisms. Let $f : M \to M$ be an Anosov diffeomorphism of a compact manifold M. (See [13].) The diffeomorphism $f^1 : F^1(M) \to F^1(M)$ naturally induced by f generates a $\mathbb{Z}$-action on $F^1(M)$ and it can be shown (from the definition of an Anosov map) that this is a proper action. Set $V = F^1(M)/\mathbb{Z}$ and let $\mathcal{G} : F^1(M) \to V$ be the natural projection. Then $\mathcal{G}$ is a smooth geometric structure, but not of algebraic type.

13. "Random" structures. Given a manifold V (or, more generally, a complete separable metric space), define the space $\mathcal{P}(V)$ of Borel probability measures on V. (A probability measure μ on V is a positive measure such that $\mu(V) = 1$.) By the Riesz representation theorem, $\mathcal{P}(V) = C_0(V)^*$ – the space of bounded linear functionals on the space of continuous functions on V that vanish at ∞. We give $\mathcal{P}(V)$ the weak*-topology: a sequence of probability measures μ_n converges to μ if for all $f \in C_0(V)$ the sequence of numbers $\int f d\mu_n$ converges to $\int f d\mu$.

Suppose now that V comes with a continuous action of G^r. There is then a continuous action of G^r on $\mathcal{P}(V)$, which is defined as follows: if $g : V \to V$

9

is any homeomorphism of V and $\mu \in \mathcal{P}(V)$, then $g_*\mu$ is the probability measure characterized by $\int f \, d(g_*\mu) = \int f \circ g \, d\mu$ for all $f \in C_0(V)$. We can now define a geometric structure of type $\mathcal{P}(V)$ on a manifold M as a G^r-equivariant (continuous) map $\mathcal{G} : F^r(M) \to \mathcal{P}(V)$.

If V is as in example 6, then $\mathcal{G}$ would assign in a continuous way to each $x \in M$ a probability measure μ_x on the Grassmannian variety of T_xM. As a special case, if μ_x is a point mass for each x, we return to the ordinary notion of a continuous subbundle of TM. Such "random" plane bundles play a role in Zimmer's proof of the Margulis superrigidity theorem. (See [22] and [6].)

Prolongations of geometric structures. Given a smooth geometric structure of order r on a manifold M, its s-jet, appropriately defined, may be regarded as a geometric structure of order $r + s$. This is called the *prolongation* of order s of the initial structure. The precise definition is given below. It should be noticed that the prolongation of any order of an A-structure is also an A-structure.

We denote by $J_n^s G^r$ the group of s-jets at 0 of smooth maps from a neighborhood of $0 \in \mathbb{R}^n$ into G^r. The group multiplication is defined in the natural way: $(j^s\varphi_1)_0 \cdot (j^s\varphi_2)_0 := j^s(\varphi_1\varphi_2)_0$. Notice that if $\eta = (j^s\varphi)_0 \in J_n^s G^r$ and $h = (j^s f)_0 \in G^s$, then $\eta h := j^s(\varphi \circ f)_0$ defines a right action of G^s on $J_n^s G^r$. With this action, we make the semidirect product $G^s \ltimes J_n^s G^r$, in which the group multiplication is

$$(h, \eta)(h', \eta') = (hh', \eta(\eta'h^{-1})).$$

The definition of prolongation of geometric structures will require the homomorphic embedding of G^{r+s} into $G^s \ltimes J_n^s G^r$ that we give next. Let $\tau_x : y \mapsto y - x$ be the translation in $\mathbb{R}^n$. Given $g = (j^{r+s}f)_0 \in G^{r+1}$, the map $f_x := \tau_x \circ f \circ \tau_{-f^{-1}(x)}$, for x is a small neighborhood of 0, is a local diffeomorphism of $\mathbb{R}^n$ fixing 0. Therefore, for each x one has $f^r(x) := (j^r f_x)_0 \in G^r$. Its s-jet at 0 only depends on g (the $r + s$-jet of f) and is an element of $J_n^s G^r$. We denote it by $a(g)$.

The desired homomorphism $i_{r,s} : G^{r+s} \to G^s \ltimes J_n^s G^r$ is now given by

$$i_{r,s}(g) = (\pi_s^{r+s}(g), a(g))$$

where $\pi_s^{r+s} : G^{r+s} \to G^s$ is the natural projection. That this is indeed a homomorphism follows from the fact that a satisfies

$$a(g_1 g_2) = a(g_1) \cdot [a(g_2)\pi_s^{r+s}(g_1)^{-1}].$$

10

Suppose now that V is a smooth manifold with a smooth (left) action of G^r on it. We denote the action of g on v simply as gv. Then $J_n^s V$ – the space of s-jets at 0 of smooth maps from (a neighborhood of 0 in) $\mathbb{R}^n$ into V – supports the following action of $G^s \ltimes J_n^s G^r$: for $(h, \xi) \in G^s \ltimes J_n^s G^r$ and $\eta \in J_n^s V$ define

$$(h, \xi)\eta := \xi \cdot (\eta h^{-1}).$$

(The dot operation is defined by $(j^s f)_0 \cdot (j^s v)_0 := j^s (fv)_0$ and the right action of G^s on $J_n^s V$ is defined just as the right action of G^s on $J_n^s G^r$ introduced earlier.) In combination with the homomorphism $i_{r,s}$ we obtain, by restriction, an action of G^{r+s} on $J_n^s V$.

Finally, the s-prolongation of a smooth geometric structure $\mathcal{G} : F^r(M) \to V$ is defined as $\mathcal{G}^s : F^{r+s}(M) \to J_n^s V$ such that

$$\mathcal{G}^s(\eta) := j^s(\mathcal{G} \circ \bar{\varphi}_r)_0$$

where $\eta = (j^{r+s}\varphi)_0 \in F^{r+s}(M)$ and $\bar{\varphi}_r(x) := j^r(\varphi \circ \tau_{-x})_0 \in F^r(M)$, for each $x \in \mathbb{R}^n$ sufficiently close to 0.

It can be shown that $\mathcal{G}^s$ is indeed a G^{r+s}-equivariant map. Furthermore, if $p_s : J_n^{s+1} V \to J_n^s V$ and $\pi_s : F^{s+1}(M) \to F^s(M)$ are the natural projections, then

$$\mathcal{G}^s \circ \pi_r = p_r \circ \mathcal{G}^{s+1}$$

for each s.

The first prolongation of a frame field. We saw in example 2 of the previous subsection that a frame field on M can be described by a $GL(n, \mathbb{R})$-equivariant map $\mathcal{G} : F^1(M) \to GL(n, \mathbb{R})$. The first prolongation of a smooth frame field is similarly described by a G^2-equivariant map $\mathcal{G}^1 : F^2(M) \to J_n^1 G$ $(G = GL(n, \mathbb{R}))$. There is a natural identification of $J_n^1 G$ and $G \times W$, where $W = \mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$, which uses that each tangent $T_g G$ is isomorphic to $T_e G = \mathfrak{g} = \mathbb{R}^{n*} \otimes \mathbb{R}^n$ under the derivative of the left-translation map by g^{-1}. In this way $G \times W$ is a Lie group, with multiplication given by

$$(g_1, \alpha_1)(g_2, \alpha_2) = (g_1 g_2, \mathrm{Ad}(g_2)^{-1} \circ \alpha_1 + \alpha_2).$$

($\mathrm{Ad}(g)X = gXg^{-1}$, $X \in \mathfrak{g}$.) The right-multiplication of $J_n^1 G$ by G is simply

$$((g, \alpha), h) = (g, \alpha \circ h).$$

and the semidirect product structure on $G \ltimes J_n^1 G$ corresponds to the multiplication

$$(h, (g, \alpha))(h', (g', \alpha')) = (hh', (gg', \mathrm{Ad}(g')^{-1} \circ \alpha + \alpha' \circ h^{-1})).$$

Representing an element of G^2 by the pair (h, η), where $h \in G$ and $\eta \in S^2(\mathbb{R}^{n*}) \otimes \mathbb{R}^n \subset \mathbb{R}^{n*} \otimes \mathfrak{g}$, the homomorphic imbedding of G^2 into $G \ltimes J_n^1 G$ corresponds to the map

$$(h, \eta) \mapsto (h, (h, \eta(h^{-1} \cdot, \cdot))).$$

Finally, the (left) action of G^2 on $J_n^1 G$, involved in the first prolongation of a frame field is given by

$$(h, \eta)(g, \alpha) = (hg, \mathrm{Ad}(g)^{-1} \circ \eta(h^{-1} \cdot, \cdot) + \alpha \circ h^{-1}).$$

If N is the kernel of the natural projection from G^2 onto G, it is not difficult to check that the quotient $J_n^1 G/N$ is a vector bundle over G with typical fiber $\Lambda^2(\mathbb{R}^{n*}) \otimes \mathbb{R}^n$. The map $\bar{\mathcal{G}} : F^1(M) \to J_n^1 G/N$ is a first order structure that corresponds to giving at each x the frame at x and the (antisymmetric) coefficients of the Lie brackets between vector fields in the frame field.

Prolongation of affine connections. Recall that a symmetric connection is a geometric structure with

$$V = S^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$$

and the action of G^2 on V is given by

$$(A, \alpha) \cdot \Gamma = A\Gamma(A^{-1} \cdot, A^{-1} \cdot) - \alpha(A^{-1} \cdot, A^{-1} \cdot).$$

It will be convenient to introduce (global) coordinates on G^r and $J_n^s V$. On G^r, coordinates $\beta_{j_1 \cdots j_k}^i : G^r \to \mathbb{R}$ ($1 \leq i \leq n$, $1 \leq j_1 \leq \cdots \leq j_k \leq n$) are defined by
$$\beta_{j_1 \cdots j_k}^i((j^s \varphi)_0) = (D_{j_1} \cdots D_{j_k}(\varphi^{-1})_i)(0)$$

where $(\varphi^{-1})_i$ is the i-th component of the inverse of φ, D_j denotes the partial derivative with respect to the j-th coordinate of $\mathbb{R}^n$, and $k \leq r$. Also introduce
$$\alpha_j^i((j^s \varphi)_0) = \beta_j^i((j^s \varphi^{-1})_0).$$

Define (global) coordinates $\Gamma_{j_1 j_2, k_1 \cdots k_l}^i$ on $J_n^s V$ as follows: $\Gamma_{jk}^i(v)$ is the i-th component of the vector $v(e_j, e_j)$, $v \in V$, where as always $\{e_1, \ldots, e_n\}$ stands for the canonical basis of $\mathbb{R}^n$, and

$$\Gamma_{j_1 j_2, k_1 \cdots k_l}^i((j^s v)_0) = D_{k_1} \cdots D_{k_l}(\Gamma_{j_1 j_2}^i \circ v)(0).$$

An elementary calculation shows that the action of G^3 on $J_n^1 V$ obtained by 1-prolongation of the action of G^2 on V is given in the above coordinates as follows: for $g \in G^3$ and $\xi \in J_n^1 V$,

$$\Gamma_{jk}^i(g\xi) = \alpha_p^i(\beta_j^q \beta_k^r \Gamma_{qr}^p + \beta_{jk}^p)$$
$$\Gamma_{jk,m}^i(g\xi) = -\alpha_s^i \alpha_p^t \beta_{tm}^s (\beta_j^q \beta_k^r \Gamma_{qr}^p + \beta_{jk}^p) +$$
$$+ \alpha_p^i((\beta_{jm}^q \beta_k^r + \beta_j^q \beta_{km}^r)\Gamma_{qr}^p + \beta_j^q \beta_k^r \beta_m^t \Gamma_{qr,t}^p + \beta_{jkm}^p).$$

(We are using here the usual summation convention. For simplicity, we dropped g and ξ from the right-hand side of the equations.)

In spite of its messy expression, the next proposition shows that it is possible to obtain some useful information about the action of G^3 on $J_n^1 V$. Define the functions $R_{jkm}^i : J_n^1 V \to \mathbb{R}$ by

$$R_{jkm}^i = \Gamma_{jk,m}^i - \Gamma_{jm,k}^i + \Gamma_{pm}^i \Gamma_{jk}^p - \Gamma_{pk}^i \Gamma_{jm}^p.$$

These are the components of the formal curvature tensor. If N^r denotes the kernel of the projection homomorphism $\pi_1^r : G^r \to G^1 = GL(n, \mathbb{R})$, then it is easily shown that the functions R_{jkm}^i are N^3-invariant. (This is simply an expression of the multilinearity of the curvature tensor.)

The following proposition is proved in [15].

Proposition 2.1 $J_n^s V$ is a (trivial) left principal N^{s+2}-bundle and its base $\bar{V} = N^{s+2}\backslash J_n^s V$ is diffeomorphic to $\mathbb{R}^d$, for some positive integer d. Furthermore, $GL(n, \mathbb{R})$ naturally acts on $\bar{V}$ in such a way that the bundle projection $\rho : J_n^s V \to \bar{V}$ is equivariant, that is, $\rho(g\xi) = \pi_1^{s+2}(g)\rho(\xi)$ for $g \in G^{s+2}$ and $\xi \in J_n^s V$. (This is also true for linear connections with torsion, after replacing V with $\mathbb{R}^{n*} \otimes \mathbb{R}^{n*} \otimes \mathbb{R}^n$.) Let, now, $s = 1$. Then $d = n^2(n^2 - 1)/3$ and if $\bar{R}_{jkm}^i$ are the functions on $\bar{V}$ defined by $R_{jkm}^i = \bar{R}_{jkm}^i \circ \rho$, then from among the $\bar{R}_{jkm}^i$ one can extract a coordinate system for $\bar{V}$.

By counting dimensions, it follows from Proposition 2.1 that the well-known symmetries of the (formal) curvature tensor

$$R_{jkm}^i + R_{jmk}^i = 0, \quad R_{jkm}^i + R_{mjk}^i + R_{kmj}^i = 0$$

are the only dependencies among these functions.

3 Isometries

A diffeomorphism f of the smooth manifold M induces a diffeomorphism f^r of $F^r(M)$ by setting $f^r((j^r\varphi)_0) := j^r(f \circ \varphi)_0$. If $g \in G^r$ and $\xi \in F^r(M)$,

then clearly $f^r(\xi g) = f^r(\xi) g$, so that f^r is an automorphism of the principal G^r-bundle $F^r(M)$.

Let $\mathcal{G} : F^r(M) \to V$ be a smooth geometric structure on M of type V and order r. A diffeomorphism $f : M \to M$ is said to be an *isometry* of $\mathcal{G}$ if

$$\mathcal{G} \circ f^r = \mathcal{G}.$$

The collection of all isometries of a geometric structure $\mathcal{G}$ on M forms a group, denoted $\mathrm{Iso}(M, \mathcal{G})$. This is a topological group, with the compact-open topology.

Take as an example a pseudo-Riemannian metric on M (example 4). We remark that $f^1(\xi) = df_x \circ \xi$, where $df_x : T_x M \to T_{f(x)} M$ is the tangent map associated to f. Then the condition $\mathcal{G} \circ f^1 = \mathcal{G}$ is easily seen to be equivalent to

$$\langle v, w \rangle_{f(x)} = \langle (df_x)^{-1} v, (df_x)^{-1} w \rangle_x.$$

(We are using the notations of example 4.) This is the ordinary notion of isometry for a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$.

Local and infinitesimal isometries. If U is an open subset of M, we will denote by either $F^r(M)|_U$ or $F^r(U)$ the preimage of U under the projection $\pi : F^r(M) \to M$. A diffeomorphism $f : U_1 \to U_2$ between open sets U_1, U_2 is called a *local isometry* of $\mathcal{G}$ if

$$\mathcal{G}|_{F^r(U_1)} \circ f^r = \mathcal{G}|_{F^r(U_2)}.$$

The collection of all local isometries of $\mathcal{G}$ forms a pseudo-group, which will be denoted $\mathrm{Iso}^{\mathrm{loc}}(M, \mathcal{G})$. This pseudo-group can be given the compact-open topology, which is the topology generated by a base whose sets have the form

$$N_1(K, U) = \{ \Phi \in \mathrm{Iso}^{\mathrm{loc}}(M, \mathcal{G}) \mid K \subset \mathrm{dom}(\Phi), \Phi(K) \subset U \}$$
$$N_2(U, K) = \{ \Phi \in \mathrm{Iso}^{\mathrm{loc}}(M, \mathcal{G}) \mid U \subset \mathrm{dom}(\Phi), K \subset \Phi(U) \}$$

where $K \subset M$ is compact, $U \subset M$ is open and $\mathrm{dom}(\Phi)$ denotes the domain of Φ. (We also use later $\mathrm{Im}(\Phi)$ for the image set.) (See [18].)

We can also define a C^k-topology on any pseudo-group $\mathcal{P}$ of smooth local diffeomorphisms as the compact-open topology of the prolongation $\mathcal{P}^k$ of $\mathcal{P}$, which is the pseudo-group of local diffeomorphisms of $F^{r+k}(M)$ of the form $\Phi^k : F^{r+k}(\mathrm{dom}(\Phi)) \to F^{r+k}(\mathrm{Im}(\Phi))$.

A *germ* of isometry from x to y is an equivalence class of local isometries sending x to y under the relation that identifies two local isometries if and only if they coincide on some open set that contains x and is contained

in the intersection of their domains. The collection of germs of isometries forms a groupoid, which will be denoted by $\mathrm{Iso}^{\mathrm{germ}}(M, \mathcal{G})$. It also has a C^k-topology, defined as follows. Let $\mathcal{P}$ be any pseudo-group of smooth local diffeomorphism of M and define the surjective map

$$E : \{(x, \Phi) \in M \times \mathcal{P} \mid x \in \mathrm{dom}(\Phi)\} \to \mathcal{P}^{\mathrm{germ}}$$

that evaluates the germ at x represented by Φ. Then $\mathcal{P}^{\mathrm{germ}}$ is given the *identification topology*, whose open sets are U such that $E^{-1}(U)$ is open.

Finally, we define an *infinitesimal isometry* or order s from x to y as the s-jet of a local diffeomorphism from a neighborhood of x to a neighborhood of y, sending x to y, such that for each $\xi \in F^{r+s}(M)$ above x

$$\mathcal{G}^s(f^{r+s}(\xi)) = \mathcal{G}^s(\xi).$$

The collection of infinitesimal isometries of order k forms a groupoid, denoted by $\mathrm{Iso}^k(M, \mathcal{G})$.

Let $D^s(M)$ denote the space of s-jets of local diffeomorphisms of M. $D^s(M)$ is naturally identified with the quotient $(F^s(M) \times F^s(M))/G^s$ where G acts on the product by $(\xi_1, \xi_2)g = (\xi_1 g, \xi_2 g)$. The identification is achieved by the map

$$(\xi_1, \xi_2)G^s \mapsto j^s(\varphi_2 \circ \varphi_1^{-1})_0$$

where $\xi_1 = (j^s \varphi_1)_0$ and $\xi_2 = (j^s \varphi_2)_0$. This shows that $D^s(M)$ is a smooth manifold diffeomorphic to a locally trivial fibration over $M \times M$ with fibers diffeomorphic to G^s. The fiber above (x, y) will be denoted $D^s(M)_{x,y}$ and comprises the s-jets (at x) of local diffeomorphisms of M sending x to y. The natural projection $p : F^s(M) \times F^s(M) \to D^s(M)$ is easily seen to be a principal G^s-bundle.

The symbol $\mathrm{Iso}^{\mathrm{germ}}_{x,y} = \mathrm{Iso}^{\mathrm{germ}}_{x,y}(M, \mathcal{G})$ stands for the germs of isometries taking x to y. $\mathrm{Iso}^k_{x,y} \subset D^{r+k}(M)_{x,y}$ is defined similarly. Notice that the natural projections between jet spaces yield maps

$$\mathrm{Iso}^{\mathrm{loc}}_{x,y} := \{\Phi \in \mathrm{Iso}^{\mathrm{loc}} \mid x \in \mathrm{dom}(\Phi), \Phi(x) = y\} \to \mathrm{Iso}^{\mathrm{germ}}_{x,y} \to \mathrm{Iso}^{k+1}_{x,y} \to \mathrm{Iso}^k_{x,y}.$$

$\mathrm{Iso}^k(M, \mathcal{G})$ determines an equivalence relation on M by the condition

$$x \sim y \Leftrightarrow \mathrm{Iso}^k_{x,y}(M, \mathcal{G}) \neq \varnothing.$$

The graph of this equivalence relation (a subset of $M \times M$) will be denoted $\mathcal{R}^k$.

Since $\mathcal{G}^k : F^{r+k}(M) \to J_n^k V$ is a G^{r+k}-equivariant map, it gives rise to a continuous map between the quotient spaces (with the quotient topology):

$$\bar{\mathcal{G}}^k : M = F^{r+k}(M) \to J_n^k V/G^{r+k}.$$

Notice that $\bar{\mathcal{G}}^k$ is invariant under (local) isometries of $\mathcal{G}$, that is $\bar{\mathcal{G}}^k \circ f = \bar{\mathcal{G}}^k$ if f is an isometry.

The identification $D^s(M) = (F^s(M) \times F^s(M))/G^s$ is assumed in the next proposition.

Proposition 3.1 Set $s = r + k$ and suppose that $\mathcal{G}$ is a smooth geometric structure over M of order r and type V. Then, $\mathrm{Iso}^k(M, \mathcal{G})$ is characterized by

$$p^{-1}(\mathrm{Iso}^k(M, \mathcal{G})) = (\mathcal{G}^k \times \mathcal{G}^k)^{-1}(\Delta)$$

where Δ is the diagonal in $J_n^k V \times J_n^k V$. Furthermore, the orbits of the Iso^k-equivalence relation are exactly the level sets of the function $\bar{\mathcal{G}}^k$, that is

$$\mathrm{Iso}_{x,y}^k(\mathcal{G}, M) \neq \varnothing \Longleftrightarrow \bar{\mathcal{G}}^k(x) = \bar{\mathcal{G}}^k(y).$$

Proof. This is a trivial consequence of the definitions. $\square$

Killing fields. A smooth vector field X on M gives rise to a local flow Φ_t on M, which in turn induces a local flow Φ_t^r on $F^r(M)$. The infinitesimal generator of Φ_t^r is a vector field on $F^r(M)$ that commutes with the action of G^r, since this is the case for the induced local flow. The resulting vector field is a lifting of X to $F^r(M)$. It will be denoted by X^r. We say that X is a *Killing field* for a geometric structure $\mathcal{G}$ if Φ_t is an isometry of $\mathcal{G}$ for each t. This is equivalent to $X^r(\xi)$ being in the kernel of $d\mathcal{G}_\xi$ for each $\xi \in F^r(M)$.

We can also define infinitesimal Killing fields of order i, as follows. First observe that for each $\xi \in F^{r+i}(M)$ in the fiber of some $x \in M$ there is a canonical linear isomorphism between $T_\xi F^{r+i}(M)$ and the space of r-jets of local vector fields on M around x. The latter space will be denoted by $J_x^{r+i}TM$. In fact, let X be a vector field on M representing an element of $J_x^{r+i}TM$. The vector $X^{r+i}(\xi) \in T_\xi F^{r+i}(M)$ can be shown to depend only on $j^{r+i}X_x$. We say that $\alpha \in J_x^{r+i}TM$ is an infinitesimal Killing field of order i of a geometric structure $\mathcal{G}$ or order r if, under the identification $T_\xi F^{r+i}(M) \cong J_x^{r+i}TM$, we have $d\mathcal{G}_\xi \alpha = 0$.

Complete parallelism. A simple, but for our purposes important, example of geometric structure is a complete parallelism. Recall that this is simply a section σ of $F^1(M)$. In a local coordinate system of M with coordinate vector fields $X_i = \frac{\partial}{\partial x^i}$ one has $\sigma(x) = \sum_{ij} \theta(x)_j^i du^j \otimes X_i$, where

$\Theta = (\theta^i_j)$ is an invertible matrix and u^i are the standard coordinates of $\mathbb{R}^n$. The inverse of Θ will be denoted $\Lambda = (\lambda^i_j)$. The condition for a vector field X to be a Killing field of σ amounts to $\mathcal{L}_X\sigma = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to X. A simple calculation shows that if $X = \sum f^i X_i$ then the functions f^i must satisfy the system of differential equations:

$$\frac{\partial f^k}{\partial x^l} = \sum_i A^k_{li} f^i, \quad A^k_{li} := \sum_l \lambda^s_l \frac{\partial \theta^k_s}{\partial x^i}.$$

If $c(t)$ is a differentiable path contained in the coordinate neighborhood of M on which the previous system of equations holds, $c'(t) = v^i(t)X_i$ is its velocity vector field and $h^i(t) := f^i(c(t))$, then the h^i satisfy along c the equation:

$$\frac{dh^k}{dt} = \sum_i \mathcal{A}^k_i h^i, \quad \mathcal{A}^k_i := \sum_l A^k_{li} v^l$$

Using the uniqueness of solutions of ordinary differential equations we immediately get the next proposition.

Proposition 3.2 Suppose that M is connected. If X, Y are Killing vector fields of a (differentiable) complete parallelism of M such that $X(x) = Y(x)$ for some $x \in M$, then $X = Y$.

For later use, we register here the following fact about Killing fields of a real analytic parallelism. It is taken from [1].

Proposition 3.3 Let M be a connected, simply connected analytic manifold and suppose that σ is a real analytic complete parallelism on M. Then any locally defined Killing field of σ has a unique extension to a global Killing field.

Proof. The uniqueness of the extension has already been shown. (In any event, if any two analytic vector fields of a connected manifold agree on a nonempty open set they must coincide everywhere.) By a standard argument of analytic continuation, it suffices to prove that if $\varphi : U_0 \to U$ is an analytic parametrization of a coordinate neighborhood U of $x \in M$ such that U_0 is a convex open subset of $\mathbb{R}^n$, then any Killing field defined on a connected nonempty open subset of U has a (unique) extension to U.

By transferring the problem to $\mathbb{R}^n$ via the parametrization φ, we may assume that $0 \in V$ and $U_0 = U$. For each $x \in U$ define a curve $c_x : [0, 1] \to U$ by $c_x(t) = tx$. The map $U \times [0, 1] \to U$ defined by $(x, t) \mapsto c_x(t)$ is, of course,

analytic. It was seen earlier that the differential equation for components of the Killing field along c_x is

$$\frac{dh^k}{dt} = \sum_i \mathcal{A}_i^k h^i$$

where $\mathcal{A}_i^k := \mathcal{A}_i^k(x,t)$ is an analytic function jointly on x and t. ($\mathcal{A}$ depends on x due to the dependence of c_x on x.) Therefore, we have a linear system of differential equations depending analytically on a parameter x. A solution $Y_x(t) = (w_x^1(t), \ldots, w_x^n(t))$ with initial condition $h^i(0) = v^i$, where v^i is the i-th component of X at 0, exists over the whole interval $[0,1]$. The analytic vector field $Y(x) = Y_x(1)$, $x \in U$, is the extension of X we seek. $\qquad\square$

Cartan's structures of finite type. Let $\mathcal{G}$ be an L-structure of order r. Recall that it can be defined by a reduction $P \subset F^r(M)$, hence P is a principal L-bundle. P is related to a sequence of structures of order 1

$$P \to P_{-1} \to \cdots \to P_{-r} = M$$

where P_{-i} is the projection of P into $F^{r-i}(M)$. By this remark, the study of L-structures can be reduced to the study of L-structures of order 1. With this in mind we assume in this subsection that $\pi : P \to M$ is a subbundle of $F^1(M)$.

Let θ be the canonical form on P. It is a one form on P taking values in $\mathbb{R}^n$ defined, for each $X \in T_\xi P$, by

$$\theta_\xi(X) := \xi^{-1} d\pi_\xi X.$$

(Recall that $\xi \in F^1(M)$ is viewed as an isomorphism from $\mathbb{R}^n$ to $T_x M$, $x = \pi(\xi)$.)

Let $\mathfrak{l}$ be the Lie algebra of L. Define a linear map $\partial : \mathfrak{l} \otimes \mathbb{R}^{n*} \to \mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n*}$ by

$$(\partial f)(u_1, u_2) := -f(u_2)u_1 + f(u_1)u_2.$$

Define a Lie group L_1, called the *first prolongation of L*, to be the subgroup of linear transformations T of $\mathbb{R}^n \oplus \mathfrak{l}$ such that $TX = X$ for $X \in \mathfrak{l}$ and, for some h in the image of ∂, $T(u) = u + h(\cdot, u)$, $u \in \mathbb{R}^n$. It can be shown that L_1 is (isomorphic to) a subgroup of N^2 – the kernel of the projection homomorphism from G^2 onto $G^1 = GL(n, \mathbb{R}^n)$.

Fix a linear subspace C of $\mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n*}$ such that

$$\mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n*} = C \oplus \partial(\mathfrak{l} \otimes \mathbb{R}^{n*}).$$

We define a principal L_1-bundle P_1 over P as follows. P_1 will be a L_1-reduction of $F^1(P)$, hence each element of P_1 above $\xi \in P$ is a linear frame on $T_\xi P$. A choice of a horizontal subspace H of $T_\xi P$, that is, such that $\theta|_H : H \to T_x M$ is a linear isomorphism, specifies a frame on $T_\xi P$. (The frame ξ of $T_x M$ lifts to a frame on H and the tangent space at ξ to the fiber above x has a frame induced by the principal action of L.) Define $c(\xi, H) \in \mathbb{R}^n \otimes \Lambda^2 \mathbb{R}^{n*}$ as follows: if X_1 and X_2 are the unique elements of H such that $\theta(X_i) = v_i$, for given $v_1, v_2 \in \mathbb{R}^n$, then

$$c(\xi, H)(v_1, v_2) = d\theta(X_1, X_2).$$

It is not hard to show (see [14]) that if H' is another horizontal subspace at ξ, then $c(\xi, H') - c(\xi, H)$ lies in the image of ∂. We can now define P_1: the elements of P_1 that map to $\xi \in P$ are those frames in $F^1(P)$ obtained from a horizontal subspace H such that $c(\xi, H)$ lies in C. (Therefore, the definition depends on the choice of C.) It can be shown that P_1 is, indeed, a principal L_1-bundle over P. (The reader is again referred to [14] for the details.)

The k-th prolongation of P is defined inductively: $P_k = (P_{k-1})_1$. P is said to be a *structure of finite type* (equal to k) if L_k is trivial for some k (and L_{k-1} is not). In this case, $\pi_{k-1}^k : P_k \to P_{k-1}$ is an isomorphism of principal bundles. It follows from the definition that if P is a structure of finite type (k) then the $k-1$-prolongation has a complete parallelism.

It also follows from the definition that if $\varphi : M \to M$ is an isometry of P (so that $\varphi^1 : F^1(M) \to F^1(M)$ restricts to an automorphism of P) then φ^1 induces in a canonical (and obvious) way an automorphism of P_i for each i. With these remarks and the earlier discussion about Killing fields of complete parallelism one obtains the next result from [1].

Theorem 3.4 (Amores) Let M be a connected simply connected analytic manifold, and $\mathcal{G}$ an analytic L-structure on M of finite type. If U is an open connected nonempty subset of M and X is a local Killing field of $\mathcal{G}$ defined on U, then X has a unique extension to a (globally defined) Killing field over M.

Proof. The missing details, which by now are not many, can be found in [1]. $\square$

Examples of L-structures of finite type are: pseudo-Riemannian metrics, affine connections, projective structures, (pseudo-Riemannian) conformal structures in dimension greater than 2.

Proposition 3.5 Let $\mathcal{G}$ be an L-structure of finite type k. Then, for each $x \in M$, the projection homomorphism $\mathrm{Iso}_{x,x}^k(M,\mathcal{G}) \to \mathrm{Iso}_{x,x}^{k-1}(M,\mathcal{G})$ is injective.

Proof. This is a straightforward consequence of the definitions. $\qquad\square$

4 Rigid structures.

In [20], I. M. Singer asks the following question: How alike must two points of a Riemannian manifold be in order to conclude that the manifold is homogeneous? For there to be a local isometry between neighborhoods of two points of the manifold, the curvature and its covariant derivatives at the two points must clearly be the same. The following converse is proved in [20]: if the n-dimensional Riemannian manifold M is infinitesimally homogeneous, that is, if the curvature and its k covariant derivatives (for $k < n(n-1)/2$) coincide at any two points then M is locally homogeneous, that is, there is a local isometry between neighborhoods of any two points, and if M is complete and simply connected then M is homogeneous.

A partial, but far reaching, generalization of Singer's result is obtained by M. Gromov in [10]. The result applies to a broad class of geometric structures and has been used in a number of investigations connecting geometry and dynamics. (See [3] and [4] for a particularly striking example concerning the classification of Anosov diffeomorphisms and flows.)

Gromov's result applies to a large class of geometric structures that includes pseudo-Riemannian metrics, affine connections, and Cartan's structures of finite type, which he calls *rigid A-structures*. It will be shown that if M is a smooth manifold and $\mathcal{G}$ is a smooth rigid A-structure on M, then there exists an open dense subset $U \subset M$ such that any "infinitesimal isometry" (sufficiently close to the identity) at a point in U is the jet of a local isometry.

Let $\mathcal{G}$ be a smooth geometric A-structure of order r on a manifold M. Then $\mathcal{G}$ is said to be *rigid* (or $r+i$-rigid) if the homomorphism $\mathrm{Iso}_{x,x}^{i+1} \to \mathrm{Iso}_{x,x}^i$ is injective for all $x \in M$.

Here are some examples of rigid structures.

Immersions. A C^1 0-order structure of type V is simply a C^1 map $\mathcal{G}$ from M into V. If $\mathcal{G}$ is 0-rigid, and f is local diffeomorphism of M fixing x whose first jet at x is an infinitesimal isometry of order 1, then f must be

the identity up to first jet. But f must satisfy

$$d(\mathcal{G} \circ f)_x = d\mathcal{G}_x$$

so that $\mathcal{G}$ is 0-rigid if and only if $\mathcal{G}$ is an immersion.

Anosov map. This refers to example 12 of section 2. φ is an infinitesimal isometry of $\mathcal{G}$ of order i exactly when it is the $i+1$-jet of some iterate of the Anosov map f. In particular, it is 0-rigid.

L-structures of finite type. These are rigid, according to Proposition 3.5. In particular, complete parallelisms, affine connections, pseudo-Riemannian metrics are all examples of rigid structures. A direct proof that complete parallelisms are rigid is rather easy. One can also show that an affine connection is rigid by noting that its horizontal distribution on $F^1(M)$ determines a parallelism on $F^1(M)$. As for pseudo-Riemannian metrics, it suffices to remember that they produce a connection in a canonical way – the Levi-Civita connection.

Pseudo-Riemannian conformal structures in dimension at least 3, projective structures, homogeneous structures of sufficiently high order, Itô vector fields, are also rigid structures. On the other hand, k-plane distributions, vector fields, symplectic forms, conformal structures in dimension 2 are not.

Proposition 4.1 If $\mathcal{G}$ is $r + i$-rigid then it is also $r + i + 1$-rigid.

Proof. There is no loss of generality in setting $i = 0$. By choosing ξ in $F^{r+1}(M)$ in the fiber above $x \in M$, $\mathrm{Iso}^1_{x,x}$ can be identified with the subgroup $\mathcal{I}$ of G^{r+1} consisting of all g such that $g \cdot v_0 = v_0$, where $v_0 := \mathcal{G}^1(\xi)$. Let $N := N^{r+1}$ denote the kernel of the projection homomorphism from G^{r+1} onto G^r. Then the structure is r-rigid if and only if $\mathcal{I} \cap N$ is the trivial group.

Fix $\xi' \in F^{r+2}(M)$ projecting to ξ and let $\mathcal{I}'$ be the subgroup of G^{r+2} that corresponds to $\mathrm{Iso}^2_{x,x}$ under the choice of ξ'. We also define the notations: $v'_0 := \mathcal{G}^2(\xi')$ and $N' := N^{r+2}$. The claim amounts to proving that if $\mathcal{I} \cap N$ is trivial then $\mathcal{I}' \cap N'$ is also trivial.

Before continuing, the following remarks will be needed. First, the action of G^{r+1} on $J_n^1 V$ will be written as $\rho(g, v)$ rather than simply as gv. Second, we point out that if N is any manifold, then $J_n^1 N$ is naturally identified with the bundle $\mathbb{R}^{n*} \otimes TN$ over N. Third, it is not difficult to show that, under the homomorphism $i_{r+1,1} : G^{r+2} \to G^1 \ltimes J_n^1 G^{r+1}$ described earlier in the text, the subgroup N' maps into $\{e\} \times J_n^1 N$, where $e = e^1$ is the identity in G^1.

Let g is a smooth map from a neighborhood of $0 \in \mathbb{R}^n$ into N whose differential dg_0 describes $g' \in \mathcal{I}'$, and let v be a smooth map from another neighborhood of 0 into $J_n^1 N$ such that dv_0 corresponds to v_0', hence $v(0) = v_0$. The element $g' \cdot v_0'$ then corresponds to the differential of $\rho(g, v)$ at 0. The condition $g' \cdot v_0' = v_0'$ corresponds to $d\rho(g, v)_0 = dv_0$; in particular $g(0) \cdot v_0 = v_0$, so that $g(0)$ is the identity $e = e^{r+1} \in N$. With these notations and remarks, we want to show that $dg_0 u = 0$ for each $u \in \mathbb{R}^n$.

Write $X_u := dg_0 u \in T_e N$. Then X_u is in the Lie algebra of N. We write for each $w \in J_n^1 V$ the vector $\bar{X}_u(w) \in T_v V$ given by

$$\bar{X}_u(w) := \frac{d}{dt}|_{t=0} \rho(\exp(tX_u), w).$$

$\bar{X}_u$ is a smooth vector field on V, whose flow is $\Phi_t(w) = \rho(\exp(tX_u), w)$. By the chain rule we obtain

$$d\rho(g, v)_0 u = \bar{X}_u(v_0) + dv_0 u.$$

But since $d\rho(g, v)_0 = dv_0$, we conclude that $\bar{X}_u(v_0) = 0$. As a consequence, $\rho(\exp(tX_u), v_0) = v_0$ for all t. This means that $\exp(tX_u) \in \mathcal{I} \cap \mathcal{N}$, hence $\exp(tX_u)$ is the identity element for each t. In particular, $X_u = 0$, which is what we wanted to prove. $\qquad\square$

Corollary 4.2 If $\mathcal{G}$ is a rigid A-structure on M, then for each $x \in M$ there exists $s_1 = s_1(x)$ such that $\mathrm{Iso}_{x,x}^{s+1} \to \mathrm{Iso}_{x,x}^s$ is an isomorphism for each $s \geq s_1$.

Proof. The key point here is that, $\mathcal{G}$ being an A-structure, the groups $\mathrm{Iso}_{x,x}^k$ are real algebraic groups. The corollary follows from the descending chain condition applied to algebraic groups. $\qquad\square$

Framed definition of rigidity. We have seen earlier that an L-structure of finite type produces a frame field (a complete parallelism) on P_k, for some k. The following alternative definition of rigid structures is offered in [5]: a structure $\mathcal{G}$ is k-rigid if there exists a full frame field on $F^k(M)$ obtained from $\mathcal{G}$ by a "canonical" procedure (so that an isometry of $\mathcal{G}$ induces an isometry of the full-frame). It is claimed in [5] that "rigidity" implies "framed rigidity."

The Iso-relations for rigid structures. Here and in the next few sections we study the general properties of the equivalence relation defined

by Iso^i and $\mathrm{Iso}^{\mathrm{loc}}$. The main result is described in the next theorem. (This is stated in a somewhat more restricted form than in [10]. See also [2].) The theorem is Gromov's partial generalization of Singer's theorem. A detailed proof will be given in the next section.

Theorem 4.3 (Gromov's open-dense theorem) Let M be a smooth manifold and $\mathcal{G}$ a smooth rigid A-structure on M of order r. Then, there exists a positive integer s_0 and for each $s \geq s_0$ there is an open subset $W_s \subset M \times M$ such that the following holds:

1. The equivalence classes of the Iso^s-relation are closed smooth submanifolds of W_s.

2. For each $(x, y) \in W_s$ and each $\xi \in \mathrm{Iso}^s_{x,y}$, there exists a unique (germ of) a local isometry of $\mathcal{G}$ that sends x to y. Furthermore, the correspondence from infinitesimal to local isometries is continuous.

3. The set U_s of all $x \in M$ for which $(x, y) \in W_s$ for some $y \in M$ is open dense, and the collection of $y \in M$ such that $(x, y) \in W_s$, for each $x \in U_s$, is an open neighborhood of x.

Corollary 4.4 Under the assumptions of the previous theorem, and the further assumption that $\mathrm{Iso}^s_{x,y} \neq \varnothing$ for some big enough s and all $x, y \in M$ (in other words, Iso^s is transitive for some big enough s), then every $x \in M$ has a neighborhood U_x such that $\mathrm{Iso}^{\mathrm{loc}}_{x,y}$ is non-empty for all $y \in U_x$. In particular, if the group of isometries of $(M, \mathcal{G})$ has a dense orbit in M, then $(M, \mathcal{G})$ is locally homogeneous.

Proof. By "locally homogeneous" it is meant that $\mathrm{Iso}^{\mathrm{loc}}_{x,y}$ is non-empty for all pairs $x, y \in M$. This is actually a corollary of the proof rather than the theorem itself. It will be seen in the proof of the Theorem 4.3 that the set U_s is characterized by the property that, above U_s, $\mathcal{G}^s$ and $\bar{\mathcal{G}}^s$ have locally constant rank. But the existence of an element in $\mathrm{Iso}^{i+1}_{x,y}$ implies that the rank of $\mathcal{G}^i$ is the same at $\xi, \eta \in F^{r+i}(M)$ for any ξ (respectively η) in the fiber of x (respectively y). Similarly for $\bar{\mathcal{G}}^i$. $\qquad\square$

It should be noticed that the condition that $\mathcal{G}$ be an A-structure is essential. In example 12 of section 2, $\mathcal{G}$ is rigid but not of algebraic type, and in that case the orbits of the pseudo-group of local isometries are precisely the orbits of the $\mathbb{Z}$-action generated by the Anosov element, which are typically dense but never open.

Corollary 4.5 If $\mathcal{G}$ is a rigid A-structure on M, then $\mathrm{Iso}(M, \mathcal{G})$ is a Lie group such that the natural action on M is smooth. The Lie algebra of $\mathrm{Iso}(M, \mathcal{G})$ is the space of Killing fields.

Proof. Fix $x \in U_s$ ($s \geq s_0$, as in the theorem) and consider the smooth manifold $\mathcal{I}^s := \mathrm{Iso}^s_{x, U_x}$ comprised of the infinitesimal isometries of order s sending x into U_x. This is homeomorphic to $\mathcal{I}^{\mathrm{loc}} := \mathrm{Iso}^{\mathrm{loc}}_{x, U_x}$ – the local isometries sending x into U_x. On the other hand, a neighborhood of the identity in $\mathrm{Iso}(M, \mathcal{G})$ (in the compact open topology) embeds into $\mathrm{Iso}^{\mathrm{loc}}_{x, U_x}$ with locally closed image. Therefore, $\mathrm{Iso}(M, \mathcal{G})$ has a manifold structure with dimension no greater than the dimension of $\mathcal{I}^s := \mathrm{Iso}^s_{x, U_x}$. Similar neighborhoods of points other than the identity are obtained by translation in $\mathrm{Iso}(M, \mathcal{G})$. The second claim is clear. $\square$

Integrating infinitesimal isometries of frame fields. We explain in detail the process of integrating infinitesimal isometries for the special case in which the geometric structure is a frame field. (Notice that a similar result for Cartan structures of finite type are reduced to the case of frame fields by an induction argument.)

Let $X_1, \ldots, X_n$ be smooth vector fields on an n-dimensional manifold M, everywhere linearly independent. The X_i determines a smooth frame field on M. If f is a smooth local diffeomorphism of M, we say that f is an *infinitesimal isometry* of order r of the frame field if

$$f_* X_i \overset{r}{\underset{x}{\sim}} X_i.$$

for each i, where $(f_* X)_x = df_x X(f^{-1}(x))$ represents the push-forward of a vector field X.

The Lie brackets of $X_1, \ldots, X_n$ can be expressed as

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k$$

where the c_{ij}^k are smooth real functions on M.

Let φ_t be denote the local flow of X_i and define

$$\varphi^x(t_1, \ldots, t_n) := (\varphi_{t_n}^n \circ \cdots \circ \varphi_{t_n}^n)(x).$$

An application of the inverse function theorem shows that φ^x is a local parametrization of M near x, that is, there exists an open neighborhood

$U_x \subset \mathbb{R}^n$ of 0 and $V_x \subset M$ of x such that $\varphi^x : U_x \to V_x$ is a smooth diffeomorphism that sends 0 to x.

For each $x \in M$, denote by θ^x the smooth map from U_x into $\mathbb{R}^{n^3}$ defined by

$$\theta^x := \left(c_{ij}^k \circ \varphi^x \right).$$

The r-jet of θ^x at 0 is

$$\Theta^r(x) := (j^r \theta^x)_0 \in J^r(\mathbb{R}^n, \mathbb{R}^{n^3})_0.$$

Lemma 4.6 For each x and y in M, $\Theta^r(x) = \Theta^r(y)$ if and only if $\Phi := \varphi^y \circ (\varphi^x)^{-1}$ is an infinitesimal isometry of order $r + 1$ of the frame field $X_1, \ldots, X_n$ at x.

Proof. For each i, write

$$\Phi_* X_i =: \sum_{j=1}^n h_{ij} X_j$$

and introduce the matrix $H = (h_{ij})$. If ∂_k denotes the kth coordinate vector field of $\mathbb{R}^n$, then the definition of φ^x implies $(\varphi^x_* \partial_k)_x = X_k(x)$, so that H is the identity matrix at y. We need to show that all the derivatives of H of order up to $r + 1$ at y are 0.

Let S_i^x, $i = 1, \ldots, n$, be an i-dimensional submanifold of M diffeomorphic under φ^x to a neighborhood of the origin in $\mathbb{R}^i$. Hence, S^i is comprised of points of the form $\varphi^x(t^1, \ldots, t^i, 0, \ldots, 0)$. On S_i^x, we have $\varphi^x_* \partial_k = X_k$ for all $k \geq i$. Therefore, Φ sends X_k onto itself along S_i^x, for each $k \geq i$. This means, in particular, that $H(z)$ is the identity matrix for $z \in S_1^y$, so that all the derivatives of H along S_1^y of order up to $r + 1$ are 0. Proceeding by induction, suppose that all the derivatives of H along S_{l-1}^y at y vanish, $l \leq n$. We want to show the same for the derivatives along S_l^y. It will suffice for that to show that for each k, $1 \leq k \leq r + 1$, the k-th order derivative, $(X_l)^k H$, with respect to X_l has $r + 1 - k$-jet along S_{l-1}^y at y equal to 0. (Note that it is possible to set a coordinate system for a neighborhood of y in S_l^y so as to have X_l as one of the coordinate vector fields.)

Introduce the matrix $C^{(k)} := (c_{ij}^{(k)})$, where $c_{ij}^{(k)} := c_{ki}^j$. It follows from $\Phi_*[X_l, X_i] = [\Phi_* X_l, \Phi_* X_i]$ and from the fact that $\Phi_* X_l = X_l$ along S_l^y that H satisfies on a neighborhood of y in S_l^y the equation

$$X_l H = H C^{(l)} - (C^{(l)} \circ \Phi^{-1}) H.$$

The condition $\Theta^r(x) = \Theta^r(y)$ is equivalent to $C^{(l)}$ and $C^{(l)} \circ \Phi^{-1}$ having the same r-jet at y for each l. Therefore, $X_l H$ has along S_{l-1} the same r-jet as

the commutator $[H, C^{(l)}] := HC^{(l)} - C^{(l)}H$, which vanishes at y up to order $r + 1$ along S_{l-1}. Since

$$(X_l)^k H = \sum_{j=0}^{k-1} b_j^{k-1} [(X_l)^j H, (X_l)^{k-1-j} C^{(l)}]$$

(where b_j^{k-1} is the binomial coefficient) it follows by a second induction that $(X_l)^k H$ vanishes along S_{l-1}^y to order $r - k$ at y, for each k, as claimed. $\square$

Write $E := E^0 := M \times M$ and view E as a fibered manifold over M with respect to the projection onto the first factor. Let E^r be the space of r-jets of (germs of) local sections of E. Then E^r is a smooth fibered manifold over E^s, $s \leq r$, and also over M with respect to the projection onto the first factor. Note that E^r can be identified with $J^r(M, M)$. The projection maps will be denoted by $\pi_s^r : E^r \to E^s$ and $\pi^r : E^r \to M$. Each $\xi = j^{r+1} f_x \in E^{r+1}$ corresponds to an n-dimensional subspace, $L(\xi)$, of $T_{\bar\xi} E^r$, $\bar\xi = \pi_r^{r+1}(\xi)$, $n = \dim M$. $L(\xi)$ is defined as the image of $T_x M$ under the differential of $y \mapsto j^r f_y$ at x. We call $L(\xi)$ the *holonomic n-plane* associated to ξ. It is a *horizontal* subspace in the sense that it projects onto $T_x M$ under $d\pi_{\bar\xi}^r$. The correspondence $\xi \mapsto L(\xi)$ is smooth.

Define $\Phi^r(x, y) := j^r(\varphi^y \circ (\varphi^x)^{-1})_x$. (This should more properly be defined as $j^r(\mathrm{id}, \varphi^y \circ (\varphi^x)^{-1})_x$, but we will continue to identify the section of E with the map from M into itself defined by the section.) Then $\Phi^r(E)$ is a smooth submanifold of E^r diffeomorphic to E under π_0^r. Also introduce the set Ω^r consisting of $x \in M$ such that the rank of $d\Theta_x^r$ is locally maximal. Note that Ω^r is an open dense subset of M and the kernel $W^r(x)$ of $d\Theta_x^r$ has locally constant dimension on Ω^r. Moreover, for each r and $x \in M$, $W^{r+1}(x) \subset W^r(x)$ since $\Theta^r = \pi \circ \Theta^{r+1}$, where π denotes the projection from $J^{r+1}(\mathbb{R}^n, \mathbb{R}^{n^3})$ onto $J^r(\mathbb{R}^n, \mathbb{R}^{n^3})$. Set

$$\Omega := \bigcap_{r=0}^{2n+2} \Omega^r.$$

Then, Ω is also open and dense. Moreover, for each $x \in \Omega$, there exists an $r(x)$, $0 \leq r(x) \leq 2n$ and a neighborhood U of x such that

$$W^{r(x)}(y) = W^{r(x)+1}(y) = W^{r(x)+2}(y)$$

and $W^{r(x)}(y)$ has constant dimension d^r for $y \in U$.

By the local form of constant rank maps, each $x \in \Omega$ has a neighborhood B smoothly parametrized by a product $T \times P$, where P and T are open neighborhoods of 0 in $\mathbb{R}^{d^r}$ and $\mathbb{R}^{n-d^r}$, respectively, such the plaques $\{t\} \times P$, $t \in T$, correspond to the submanifolds of constant Θ^r in B. In other words, B is a foliated box for a smooth foliation consisting of the level sets of Θ^r. Note that, if $r = r(x)$, we can find for each $x \in \Omega$ a common foliation box B for both Θ^r and Θ^{r+1} and the two foliations coincide in B. It also follows that the set

$$\mathcal{E}_B^r := \{(x,y) \in B \times B \mid \Theta^r(x) = \Theta^r(y)\}$$

is a smooth embedded submanifold of $B \times B$ that maps onto B under the projection $(x,y) \mapsto x$ and that $\mathcal{E}_B^r$ coincides with $\mathcal{E}_B^{r+1}$ for $r = r(x)$. Introduce a submanifold of E^{r+2} given by

$$\mathcal{I}_B^{r+2} := \Phi^{r+2}(\mathcal{E}_B^r)$$

which is diffeomorphic to $\mathcal{E}_B^r$ under the projection π_0^r. To keep the notation uniform in what follows, we also introduce $\mathcal{E}_B^{-1} := B \times B$, $\mathcal{I}_B^1 := \Phi^1(\mathcal{E}_B^{-1})$. By the previous lemma, elements of $\mathcal{I}_B^{s+1}$ are infinitesimal isometries of order s of the frame field, for $s \geq 0$.

Note that, if $\mathcal{E}_B^{r-2} = \mathcal{E}_B^{r-1} = \mathcal{E}_B^r$ for some $r \geq 1$ (this is the case if $r - 2 = r(x)$), then the following is a sequence of diffeomorphisms, the maps being the natural jet projections:

$$\mathcal{I}_B^{r+2} \to \mathcal{I}_B^{r+1} \to \mathcal{I}_B^r \to \mathcal{E}_B^r.$$

We denote by $\eta_r : \mathcal{I}_B^r \to \mathcal{I}_B^{r+1}$ the inverse to the second map in the sequence.

Lemma 4.7 For each r and $a \in \mathcal{I}_B^r$, the holonomic n-plane $L(\eta_r(a))$ is a subspace of $T_a \mathcal{I}_B^r$.

Proof. Since the problem is local, we may assume that $M = \mathbb{R}^n$. An element $j^{r+1} f_x$ of $\mathcal{I}_B^{r+1}$, $y = f(x)$, is characterized by the property that the r-jets of $f_* X_i$ and of X_i at y coincide for each i. In other words,

$$D^\alpha (f_* X_i - X_i)_y = 0$$

for all α such that $|\alpha| \leq r$. A tangent vector in $T_a \mathcal{I}_B^r$ is a vector of the form $\frac{d}{dt}|_{t=0} j^r (f^{x(t),y(t)})_{x(t)}$, where $f^{x(t),y(t)}$ is a one-parameter family of local diffeomorphisms sending a differentiable curve $x(t) \in M$, $x(0) = x$, to $y(t)$, $y(0) = y$, such that $j^r (f^{x,y})_x = a$ and

$$\frac{d}{dt}|_{t=0} D^\alpha [(f^{x(t),y(t)})_* X_i - X_i]_{y(t)} = 0$$

for all α, $|\alpha| \leq r-1$. By the chain rule, this is satisfied by $j^r(f^{x(t),y(t)})_{x(t)} := j^r f_{x(t)}$, where $j^{r+1} f_x = \eta(a)$ and $x(t)$ is any differentiable curve on M such that $x(0) = x$ and $y(t) := f(x(t))$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

The previous discussion shows that the map $\eta_r : \mathcal{I}_B^r \to \mathcal{I}_B^{r+1}$ gives rise to an n-plane distribution on $\mathcal{I}_B^r$. It turns out that this distribution is involutive. Therefore, it corresponds to the tangent bundle of a local foliation with leaves of dimension n, and each leaf is the r-jet of a local section of E, hence a local isometry of the frame field. This is a consequence of a version of the classical Frobenius theorem, which is formulated and proved in the next section. The conclusion is the following:

Proposition 4.8 Let M be a smooth manifold and $X_1, \ldots, X_n$ a smooth frame field on M. Then there exists an open dense subset in M covered by a collection of open sets B_α, $\alpha \in I$, such that for each α and each pair $x, y \in B_\alpha$ for which one can find an infinitesimal isometry (at x) of the frame field, of order $2n + 3$, sending x to y, then there exists a unique local isometry defined on a neighborhood of x whose $2n + 4$-jet at x is that given by the infinitesimal isometry.

Analytic structures. We state here some special properties enjoyed by analytic structures. For the proof of the next theorem, the reader is referred to [10]. Notice that the second part of Theorem 4.9 is a generalization of Theorem 3.4.

Theorem 4.9 (Gromov) Suppose that M is a connected analytic manifold, and that $\mathcal{G}$ is an analytic rigid A-structure.

1. Let M be compact. Then, there exists an integer k and, for each $x \in M$, there exists a neighborhood U_x of x such that an infinitesimal isometry of order k or greater, taking x into U_x, extends to a local isometry (whose germ is uniquely determined).

2. If M is simply connected, then every local Killing field of $\mathcal{G}$ defined on a connected nonempty open set extends uniquely to a global Killing field.

Theorem 4.10 (Gromov) If $\mathcal{G}$ is a rigid analytic A-structure and M is compact and simply connected, then $\mathrm{Iso}(M, \mathcal{G})$ has finitely many connected components. The same holds for each isotropy subgroup $\mathrm{Iso}(M, \mathcal{G})_x$, $x \in M$.

5 Proof of Gromov's open-dense theorem

The main purpose of this section is to provide a proof of Theorem 4.3.

The Frobenius theorem. Let $\pi : E \to M$ be a fibered manifold over an n-dimensional manifold M. This means that π is a surjective submersion. The set of all k-jets of (local) sections of E is also a fibered manifold over M. It will be denoted by $\pi^k : J^k E \to M$. If $s : U \to E$ is a section of π (defined on some open set U), its k-jet $j^k s$ is a section of $J^k E$ over U. We identify $J^0 E$ with E and denote by $\pi_k^{k-1} : J^k E \to J^{k-1} E$ the natural projection. (See, for example, [9] for the general properties of $J^s E$.)

A section $\sigma : M \to J^s E$ (of class C^l) is said to be *holonomic* if it coincides with the s-jet of a (C^{s+l}) section of E. An n-dimensional subspace of $T_\eta(J^s E)$ is said to be holonomic if it is tangent at η to a local holonomic C^1 section. Notice that such a subspace is transverse to the tangent space of the fiber of $J^s E$ at η, hence it projects onto $T_x M$, for $x = \pi^s(\eta)$.

Let ξ be a point of $J^k E$ and u a section of E over a neighborhood of $x = \pi(\xi)$ such that $(j^k u)_x = \xi$. Let $\bar{\sigma}$ denote the section $j^{k-1} u$ of $J^{k-1} E$ over that neighborhood. Notice that the differential $d\bar{\sigma}_x : T_x M \to T_{\pi_{k-1}^k \xi} J^{k-1} E$ only depends on ξ, and not on the choice of the section u. We denote by $L(\xi)$ the image of $T_x M$ under $d\bar{\sigma}_x$. Therefore, $L(\xi)$ is a holonomic n-dimensional transverse subspace of $T_{\pi_{k-1}^k \xi} J^{k-1} E$.

The bundle $J^{r+s} E$ has a natural embedding into $J^r J^s E$ given by:

$$(j^{r+s} u)_x \mapsto j^r (j^s u)_x.$$

It follows from the definitions that if $\mathcal{R}$ is a submanifold of $J^s E$ such that $\pi^s|_{\mathcal{R}} : \mathcal{R} \to M$ is also a fiber bundle, then

$$J^k(J^l \mathcal{R}) \cap J^{k+l} J^s E = J^{k+l} \mathcal{R}.$$

A subset $\mathcal{R}$ of $J^s E$ will be called a *partial differential relation (PDR) of order s*. (See [11].) The *l-prolongation* of $\mathcal{R}$ is the subset set of $J^{s+l} E$ defined by

$$\mathcal{R}^l = J^l \mathcal{R} \cap J^{r+l} E.$$

Iso^k as a PDR. We can view the set of infinitesimal isometries as a partial differential relation, and a local isometry can be regarded as a solution of that system. This is done as follows. Let M be a smooth n-dimensional manifold and consider the trivial fiber bundle $E = M \times M \to M$ defined by the projection on the first factor. A section, $s(x) = (x, f(x))$, of this bundle is the graph of a map $f : M \to M$. The partial differential relation

$\mathcal{R}$ corresponding to Iso^k for some geometric structure on M of order r is then the set of all (x, ξ) where $\xi \in \text{Iso}^k_{x,y}$, for some $x, y \in M$. In this way, the set $\mathcal{R}^l$ corresponds to Iso^{k+l}.

We return now to a general E.

Lemma 5.1 Let $\mathcal{R} \subset J^s E$ be a PDR and consider a given $\xi \in \mathcal{R}$. Suppose that there exists a neighborhood $U \subset J^s E$ of ξ and a smooth map $\mathcal{F} : U \to \mathbb{R}^N$, having constant rank, such that $\mathcal{R} \cap U = \mathcal{F}^{-1}(0)$. (In particular, $\mathcal{R}$ is smooth near ξ.) If $\xi' \in \mathcal{R}^1$ maps to ξ under the natural projection, then $L(\xi') \subset T_\xi \mathcal{R}$.

Proof. The condition $\xi' \in J^1 \mathcal{R}$ means that $\xi' = j^1 \varphi_x$, where φ is a local section of $\mathcal{R}$ such that $\varphi(x) = \xi$. Therefore, $\mathcal{F} \circ \varphi = 0$, so that $d\mathcal{F}_\xi \circ d\varphi_x = 0$. But $L(\xi') = d\varphi_x T_x M$ and $T_\xi \mathcal{R} = \ker d\mathcal{F}_\xi$, and the claim follows. $\qquad\square$

We say that $\mathcal{R} \subset J^s E$ is C^k *complete* if the restriction to $\mathcal{R}$ of the natural projection π^s_{s-1} from $J^s E$ to $J^{s-1} E$ is a C^k diffeomorphism from $\mathcal{R}$ onto its image $\mathcal{R}_0$.

If the PDR consists of the infinitesimal isometries of order $s = r + u$, then the condition that it be complete means that each infinitesimal isometry of order $r + u - 1$ in the set $\mathcal{R}_0$ uniquely determines an infinitesimal isometry of order $r + u$, and that the correspondence is a C^k function.

Suppose that $\mathcal{R}$ is C^k complete, and let $h : \mathcal{R}_0 \to \mathcal{R}$ be the inverse map to $\pi^s_{s-1}|_\mathcal{R} : \mathcal{R} \to \mathcal{R}_0$. If $\eta \in J^{s-1} E$ and $\xi \in J^s E$ such that $\pi^s_{s-1}(\xi) = \eta$, then ξ determines a holonomic n-plane in $T_\eta J^{s-1} E$, which we denoted $L(\xi)$. Therefore, we obtain a C^k distribution of holonomic n-planes $\eta \mapsto \Delta(\eta) := L(f(\eta))$, $\eta \in \mathcal{R}_0$.

A partial differential relation $\mathcal{R}$ is said to be *consistent* if, for each $\xi \in \mathcal{R}$ there exists a local section $\sigma : U \to J^s E$, U open in M, such that $\sigma(x) \in \mathcal{R}$ for all $x \in U$, $\sigma(x_0) = \xi$ for some $x_0 \in U$, σ is differentiable at x_0 and

$$d\sigma_{x_0} T_{x_0} M = L(\xi')$$

for some $\xi' \in J^{s+1} E$ such that $\pi^{s+1}_s(\xi') = \xi$. In other words, for each $\xi \in \mathcal{R}$ there exists a local section of $J^s E$ taking values in $\mathcal{R}$, passing through ξ, and tangent at ξ to a holonomic n-plane.

Lemma 5.2 If $\mathcal{R}$ is both C^k complete and consistent, the n-plane $\Delta(\eta)$ is, for each $\eta \in \mathcal{R}_0$, tangent to $\mathcal{R}_0$. In particular, the projection

$$\pi^{s-1}|_{\mathcal{R}_0} : \mathcal{R}_0 \to M$$

is a submersion whose fibers are everywhere transverse to Δ.

Proof. First notice that for each $\xi' \in J^{s+1}E$ and $\xi = \pi_s^{s+1}(\xi') \in J^s E$, it is immediate from the definition of L that

$$(d\pi_{s-1}^s)_\xi L(\xi') = L(\xi).$$

Let σ be as in the definition of consistency and set $\sigma_0 := \pi_{s-1}^s \circ \sigma$. Completeness and the definition of Δ imply that $\Delta \circ \sigma_0 = L \circ \sigma$. Writing $\eta = \sigma(x_0)$, $\xi = \sigma(x)$, then $\Delta(\eta) = (d\pi_{s-1}^s)_\xi L(\xi')$ for each ξ' that projects to ξ under π_s^{s+1}. On the other hand, by the definition of consistency again, there exists ξ' such that $d\sigma_x T_x M = L(\xi')$, so that

$$\Delta(\eta) = (d\pi_{s_1}^s)_\xi d\sigma_x T_x M = d(\pi_{s-1}^s \circ \sigma)_x T_x M = (d\sigma_0)_x T_x M.$$

Since σ_0 is a local section of $\mathcal{R}_0$, this proves the claim. $\qquad\square$

For the next lemma we set $X = J^{s-1}E$ and $\pi = \pi^{s-1}$.

Lemma 5.3 Let $\mathcal{R}_0$ be a submanifold of $X|_U$, where U is an open subset of M, and suppose that $\pi|_{\mathcal{R}_0} : \mathcal{R}_0 \to U$ is a (trivial) fiber bundle. Let Δ be a C^1 n-plane distribution in $\mathcal{R}_0$ and suppose that there exists a C^1 map $h : \mathcal{R}_0 \to J^1 X$ such that $\pi_0^1 \circ h$ is the identity on $\mathcal{R}_0$ and

$$dh_\eta \Delta(\eta) = L(\xi')$$

for some $\xi' \in J^2 X$ such that $\pi_1^2(\xi') = h(\eta)$. Then Δ is involutive.

Proof. First notice that h takes values in $J^1 \mathcal{R}_0$ since $dh_\eta \Delta(\eta) = L(\xi')$ implies that $L(h(\eta)) = \Delta(\eta)$. Furthermore, writing $\Delta(\eta) = du_x TM_x$ for a local section u of $\mathcal{R}_0$ through η, we see that

$$L(\xi') = dh_\eta \Delta(\eta) = L(j^1(h \circ u)_x)$$

so that $\xi' = j^1(h \circ u)_x \in J^1 J^1 \mathcal{R}_0 \cap J^2 X = J^2 \mathcal{R}_0$. Therefore, there is no loss of generality in assuming that $\mathcal{R}_0 = X$ and that $X = U \times V$, where U and V are open subsets of $\mathbb{R}^n$ and $\mathbb{R}^q$, respectively.

We choose coordinates (x^i, u^j) on X adapted to the product, where (x^i) are coordinates for the base U and (u^j) are coordinates for the fiber V. To obtain coordinates for the jet bundles $J^1 X$, $J^2 X$ and $J^1 J^1 X$, first recall that 1_i stands for the multiindex vector of dimension n with entries 0 or 1, with 0 at all but the i-th entry. The multiindex $1_l + 1_k$ is defined by vector addition. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex vector, define

$$u_\alpha^j(j^s \sigma_x) = D_1^{\alpha_1} \cdots D_n^{\alpha_n}(u^j \circ \sigma)_x$$

31

where D_i indicates the partial derivative in x_i. This produces coordinates $(x^i, u^j, u^j_{1_l})$ on J^1X and $(x^i, u^j, u^j_{1_l}, u^j_{1_l+1_k})$ on J^2X. For J^1J^1X we introduce coordinates $(x^i, u^j, u^j_{1_l}; v^j_{1_l}, v^j_{1_l,1_k})$, where $v^j_{1_l}(j^1\sigma_x) = D_l(u^j \circ \sigma)_x$ and $v^j_{1_l,1_k}(j^1\sigma_x) = D_k(u^j_{1_l} \circ \sigma)_x$. We point out that an element $\eta \in J^1J^1X$ belongs to J^2X if and only if $v^j_{1_l}(\eta) = u^j_{1_l}(\eta)$ and $v^j_{1_k,1_l}(\eta)$ is symmetric in l and k.

The function h can now be written as

$$h(x^i, u^j) = (x^i, u^j, h^j_l(x, u)).$$

The n-plane distribution Δ is determined by h since $\Delta(\eta) = L(h(\eta))$. The lift of the coordinate vector fields $\frac{\partial}{\partial x^i}$ to Δ is

$$X_i := \frac{\partial}{\partial x^i} + \sum_m h^m_i(x, u) \frac{\partial}{\partial u^m}.$$

Then Δ is the linear span of the X_i and their Lie bracket is easily calculated:

$$[X_i, X_j] = \sum_m (a^m_{i,j} - a^m_{j,i}) \frac{\partial}{\partial u^m}$$

where

$$a^j_{l,k} := \frac{\partial h^j_k}{\partial x^l} + \sum_m h^m_l \frac{h^j_k}{\partial u^m}.$$

Therefore the lemma will follow once we prove that $a^j_{l,k}$ is symmetric in l and k.

We can express the condition $dh_\eta \Delta(\eta) = L(\xi')$ in the following way. Suppose that $u(x) = (x, f(x))$ is a local section of X such that $du_x T_x M = \Delta(\eta)$. From

$$h(u(x)) = (x^i, f^j(x), h^j_l(x, f(x)))$$

we obtain

$$j^1(h \circ u)_x = (x^i, f^j(x), h^j_l(x, f(x)); \frac{\partial f^j}{\partial x^l}(x), A^j_{l,k}(x))$$

where

$$A^j_{l,k}(x) := \frac{\partial h^j_l}{\partial x^k}(x, f(x)) + \sum_m \frac{\partial h^j_l}{\partial u^m}(x, f(x)) \frac{\partial f^m}{\partial x^k}(x).$$

But $j^1(h \circ u)_x = (j^2v)_x$ for some local section v of X, so that

$$h^j_l(x, f(x)) = \frac{\partial f^j}{\partial x^l}(x) \text{ and } A^j_{l,k}(x) = A^j_{k,l}(x).$$

These two sets of equations imply that $a^j_{l,k} = a^j_{k,l}$, as claimed. $\qquad\Box$

Lemma 5.4 A section of $J^k E$ is holonomic if and only if it is everywhere tangent to a holonomic n-plane

Proof. Since the problem is local we may assume that $E = \mathbb{R}^n \times \mathbb{R}^m$. A section of $J^k E$ is then given by $x \mapsto (x, \{\eta_\alpha(x)\}_\alpha)$ such that for each multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$, with $\alpha_1 + \cdots + \alpha_n \leq k$, η_α is a function from $\mathbb{R}^n$ into $\mathbb{R}^m$.

The section η is holonomic exactly when $\eta_\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \eta_0$, for each α, and it is tangent at every point to a holonomic n-plane if and only if, for each multiindex β, $\beta_1 + \cdots + \beta_n \leq k-1$, and each $i = 1, \dots, n$, $D_i \eta_\beta = \eta_{\beta+1_i}$. The claim follows from this remark by induction. $\qquad\Box$

Theorem 5.5 (Frobenius) Let $\mathcal{R}$ be a locally closed (that is, open in its closure) C^k submanifold of $J^s E$, $k \geq 1$. Suppose that $\mathcal{R}$ is C^k complete and consistent. Then, through each $\eta \in \mathcal{R}$ passes a local C^k solution of $\mathcal{R}$. The germ of solution at η is unique.

Proof. We already know that Δ is involutive. The classical Frobenius theorem implies that through each $\eta \in \mathcal{R}_0$ passes a unique (germ of) integral manifold of Δ. The restriction of π^{s-1} to that integral manifold is a submersion, hence invertible near η by the inverse function theorem. Therefore, we obtain a local section σ_0 of $J^{s-1} E$ that passes through η and is everywhere tangent to Δ. Since Δ is a distribution of holonomic planes, the previous lemma implies that the section is also holonomic. $\qquad\Box$

Orbits of the Iso^i-relations. A key remark made earlier about the orbits of the Iso^i-relations is that these orbits coincide with the level sets of the quotient map
$$\bar{\mathcal{G}}^i : M \to J_n^i V / G^{r+i}.$$
(See Proposition 3.1.) If $J_n^i V / G^{r+i}$ were a manifold, $\bar{\mathcal{G}}^i$ would be a smooth map, there would be an open and dense subset of M on which $\bar{\mathcal{G}}^i$ would have locally constant rank, and in that open set the level sets of $\bar{\mathcal{G}}^i$ would comprise a smooth foliation by closed submanifolds.

Although $J_n^i V / G^{r+i}$ may fail to be a manifold, Rosenlicht's Theorem 10.2 (appendix C) will imply that these remarks still hold on an open dense subset of M.

Lemma 5.6 Let $\mathcal{G}$ be a smooth A-structure on M of order r. Then for each $i \geq 0$ there exists an open dense subset $U_i \subset M$ on which $\bar{\mathcal{G}}^i$ is smooth and such that:

1. For each $x \in U_i$ there exists (i) a neighborhood $U_x \subset U_i$ of x and (ii) a smooth submanifold $W \subset J_n^i V$ whose image $\bar{W}$ in $J_n^i V / G^{r+i}$ carries a smooth manifold structure such that $\bar{\mathcal{G}}^i|_{U_x} : U_x \to \bar{W}$ is a smooth map.

2. $\bar{\mathcal{G}}^i$ has locally constant rank on U_i.

3. $\mathcal{G}^i$ has locally constant rank on $F^{r+i}(M)|_{U_i}$.

Proof. There is no loss of generality in assuming that $i = 0$ and $J_n^i V = V$. By Rosenlicht's theorem there exists a finite partition of V into G^r-invariant real varieties $V = V_0 \cup V_1 \cup \cdots \cup V_l$ such that, for each $0 \leq i \leq l$ the union $V_i \cup \cdots \cup V_l$ is Zariski closed in V, V_i is a Zariski open subset of $V_i \cup \cdots \cup V_l$ and $V_i \to V_i / H$ is a smooth fibration. Let i be the first index for which V_i intersects nontrivially the image of $\mathcal{G}$. Then $\mathcal{G}^{-1}(V_i)$ is a non-empty open subset of $F^r(M)$. It is also G^r-invariant since $\mathcal{G}$ is G^r-equivariant, so that there exists $U \subset M$ such that $\mathcal{G}^{-1}(V_i) = \pi^{-1}(U)$, where $\pi : F^r(M) \to M$ is the natural projection. But now the quotient map $\bar{\mathcal{G}}$ restricted to U is a smooth map into the smooth manifold V_i / H. If U is not already dense, we can apply the same argument to $\mathcal{G}$ restricted to the open subset of $F^r(M)$ that projects to the interior of the complement of U. After a finite iteration of this procedure we obtain an open dense subset of M, still denoted U, on which $\bar{\mathcal{G}}$ is smooth.

U contains an open and dense subset U' in which the rank of $\bar{\mathcal{G}}$ is locally maximal, hence locally constant. By the equivariance of $\mathcal{G}$ there is another open dense subset $U'' \subset M$ such that $\mathcal{G}$ has locally constant rank on $F^r(M)|_{U''}$. The intersection $U' \cap U''$ is the desired open dense set. $\qquad\square$

Define $\tilde{I}^i = p^{-1}(\mathrm{Iso}^i)$, where $p : F^{r+i}(M) \times F^{r+i}(M) \to D^{r+i}(M)$ is the projection and $D^{r+i}(M) = (F^{r+i}(M) \times F^{r+i}(M)) / G^{r+i}$ (cf. paragraph right before Proposition 3.1). If Δ is the diagonal in $J_n^i V$, then $(\mathcal{G}^i \times \mathcal{G}^i)^{-1}(\Delta)$ contains the diagonal in $U_i \times U_i$. On the other hand, for each $\xi \in U_i$, (ξ, ξ) has a neighborhood in $U_i \times U_i$ on which $\mathcal{G}^i \times \mathcal{G}^i$ has constant rank. Therefore, on that neighborhood $\tilde{I}^i$ is a smooth closed submanifold. Since $\tilde{I}^i$ is saturated by the orbits of p, we also have that Iso^i is smooth on a neighborhood of $j^{r+k}\mathrm{id}_x$ (the $r + k$-jet of the identity) at $x \in \bar{U}_i$.

Let $\bar{\mathcal{R}}^i$ be the graph of the Iso^i-relation in M, so that $\tilde{I}^i$ is mapped onto $\bar{\mathcal{R}}^i$ under the natural projection from $F^{r+i}(M) \times F^{r+i}(M) \to M \times M$. By

the same argument used for $\tilde{I}^i$ we conclude that $\bar{\mathcal{R}}^i$ is a smooth submanifold on a neighborhood of (x, x) in $\bar{U}_i \times \bar{U}_i$.

Lemma 5.7 Let $x \in U_i$. Then on a neighborhood of $j^{r+i}\mathrm{id}_x$ in $D^{r+i}(M)$ the projection $\mathrm{Iso}^i \to \bar{\mathcal{R}}^i$ is a submersion.

Proof. It suffices to show that the linear projection $T_{(\xi,\xi)}\tilde{I}^i \to T_{(x,x)}\bar{\mathcal{R}}^i$ is surjective, for each $\xi \in U_i$. This in turn is equivalent to the kernel of $d\mathcal{G}^i_\xi$ in $T_\xi F^{r+i}(M)$ mapping onto the kernel of $d\bar{\mathcal{G}}^i_x$ in $T_x M$. Let x be the base point of ξ and denote by $\xi \cdot G^{r+i}$ the fiber of x in $F^{r+i}(M)$. Similarly, let $G^{r+i} \cdot \mathcal{G}^i(\xi)$ denote the orbit of $\mathcal{G}^i(\xi)$ in $J^i_n V$. By the equivariance of $\mathcal{G}^i$ we have

$$d\mathcal{G}^i_\xi T_\xi(\xi \cdot G^{r+i}) = T_{\mathcal{G}^i(\xi)}(G^{r+i} \cdot \mathcal{G}^i(\xi)).$$

Let $v \in T_x M$ lie in the kernel of $d\bar{\mathcal{G}}^i_x$ and choose any $w \in T_\xi F^{r+i}(M)$ that projects onto v. The vector $d\mathcal{G}^i_\xi w \in T_{\mathcal{G}^i(\xi)}J^i_n V$ lies in the kernel of the projection $T_{\mathcal{G}^i(\xi)} \to T_{\bar{\mathcal{G}}^i(x)}(J^i_n V)/G^{r+i}$, hence $d\mathcal{G}^i_\xi w = d\mathcal{G}^i_\xi w'$ for some $w' \in T_\xi(\xi \cdot G^{r+i})$. But $z = w - w'$ still maps to v under the projection $T_\xi F^{r+i}(M) \to T_x M$, and is moreover in the kernel of $d\mathcal{G}^i_\xi$. $\qquad \square$

Lemma 5.8 Let $\mathcal{G}$ be a smooth rigid geometric A-structure of order r and type V on a smooth manifold M. Then there exists a positive integer s, an open dense $U \subset M$ and, for each $x \in U$, a neighborhood U_x of x such that each element of $\mathrm{Iso}^i(M, \mathcal{G})_{x,y}$, for $i \geq s$, $x \in U$, and $y \in U_x$, is the $r+i$-jet at x of an element of $\mathrm{Iso}^{\mathrm{loc}}(M, \mathcal{G})$ sending x to y, whose germ is uniquely determined by its $r+i$-jet. Furthermore, the correspondence is continuous.

Proof. The definition of rigid structure implies that the dimensions of the manifolds Iso^s stabilize after some s_0. That means that the projection $\mathrm{Iso}^{s+1} \to \mathrm{Iso}^s$ (over the open sets described above) can be inverted. This is what is needed to apply the version of the Frobenius theorem discussed earlier. $\qquad \square$

Theorem 4.3 is now a consequence of the previous lemmas.

6 Geometry and Dynamics

This section discusses some aspects of the interaction between a geometric structure and the dynamics of its group of isometries. The main results of

the section are due to R. Zimmer. (See, for example, [23]) We will make use of basic results in the theory of dynamical systems as well general facts concerning algebraic actions and semisimple groups. Appendices A, B, and C provide the necessary background.

The first remark is that a geometric structure $\mathcal{G}$ whose isometry group acts topologically transitively on M (that is, has a dense orbit) must be "essentially" an L-structure. This is due to the next proposition.

Proposition 6.1 Let V be a real algebraic H-space and P a principal H-bundle over a manifold M. Let $\mathcal{G} : P \to V$ be a C^r, $r \geq 0$, H-equivariant map and suppose that a group G of automorphisms of P acts topologically transitively on M. Suppose moreover that $\mathcal{G}$ is G-invariant. Then, there exists an open and dense G-invariant subset U of M such that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit, $H \cdot v_0 \subset V$, for some $v_0 \in V$. The set $\mathcal{G}^{-1}(v_0) \subset P$ is a C^r G-invariant L-reduction of P, where $L \subset H$ is the isotropy subgroup of v_0. If $H \cdot v_0$ is a closed subset of V, then $U = M$.

Proof. Suppose that $x_0 \in M$ has a dense G-orbit in M and let $\xi_0 \in P_x$ be any point in the fiber of P above x_0. Set $v_0 = \mathcal{G}(\xi_0)$ and denote by W the closure of the H-orbit of v_0 in V. Since the G-orbit of x_0 is dense in M, the $G \times H$-orbit of ξ_0 is also dense in P, and maps into $H \cdot v_0$. Therefore $\mathcal{G}$ maps P into W. By the general properties of algebraic actions (see appendix C) $H \cdot v_0$ is open in W, so $\mathcal{G}^{-1}(H \cdot v_0)$ is an open and dense subset of P. This set is saturated by H-orbits since $\mathcal{G}$ is H-equivariant, hence it is of the form $P|_U$ for some open and dense subset $U \subset M$. Moreover, U is G-invariant since $\mathcal{G}$ is itself G-invariant. If $H \cdot v_0$ is closed in V, then $W = H \cdot v_0$, so that $U = M$. Once we know that $\mathcal{G}$ maps into a single H-orbit, the remaining claims follow. $\qquad\square$

The same ideas also prove the measurable counterpart of the previous proposition. (This is a version of the *cocycle reduction lemma* of [22].)

Proposition 6.2 Let V be a real algebraic H-space and P a measurable principal H-bundle over a second countable metrizable space M. Let $\mathcal{G} : P \to V$ be a measurable H-equivariant map and suppose that a group G of automorphisms of P acts ergodically on M with respect to a quasi-invariant measure μ, and leaves $\mathcal{G}$ invariant. Then, there exists a G-invariant measurable conull subset U of M such that $\mathcal{G}$ maps $P|_U$ into a single H-orbit $H \cdot v_0$ in V. The pre-image of v_0 under $\mathcal{G}$ defines a measurable, G-invariant L-reduction of $P|_U$.

Proof. H-equivariance of $\mathcal{G}$ implies that $\mathcal{G}$ induces a G-invariant measurable map $\overline{\mathcal{G}} : M \to V/H$. The H-action on V is tame, since it is an algebraic action. It follows that $\overline{G}$ is constant a.e.; therefore $\mathcal{G}$ sends a G-invariant set $P|_U$, $\mu(M - U) = 0$, into a single orbit in V. $\qquad\qquad\qquad\square$

Corollary 6.3 Let $\mathcal{G} : F^r(M) \to V$ be a continuous geometric structure on M and suppose that $\mathrm{Iso}(M, \mathcal{G})$ has a dense orbit in M. Then, over an open dense $\mathrm{Iso}(M, \mathcal{G})$-invariant subset $U \subset M$, $\mathcal{G}$ is an L-reduction. A similar result holds in the measurable case.

The algebraic hull. Suppose that a group G acts by automorphisms of a principal H-bundle P and that the G-action on the base M preserves a measure class represented by a probability measure μ. We say that $Q \subset P$ is a G-invariant *measurable L-reduction* of P if Q is a measurable L-reduction of $P|_U$, for some G-invariant, μ-conull, measurable subset $U \subset M$ and the G-action on P restricts to a G-action on Q.

Proposition 6.4 (Zimmer) Let M be a second countable metrizable G-space with a quasi-invariant probability measure μ. Suppose that the action is ergodic with respect to μ. Let H be a real algebraic group and let P be a measurable principal H-bundle on which G acts by bundle automorphisms over the G-action on M. Then:

1. There exists a real algebraic subgroup $L \subset H$ and a G-invariant measurable L-reduction $Q \subset P$ such that Q is minimal; i.e., Q does not admit a measurable G-invariant L'-reduction for a proper real algebraic subgroup L' of L.

2. If Q_1 and Q_2 are G-invariant reductions with groups L_1 and L_2 resp., satisfying the above minimality property, then there is an $h \in H$ such that $L' = hLh^{-1}$ and $Q_2 = Q_1 h^{-1}$.

3. Any G-invariant measurable L'-reduction of P, for real algebraic L', contains a G-invariant measurable L''-reduction, where L'' is a conjugate in H of the minimal L obtained in item 1.

Proof. Let $Q_1 \supset Q_2 \supset \cdots$ be a nested sequence of invariant reductions with groups $L_1 \supset L_2 \supset \cdots$. The groups L_i form a descending chain of real algebraic groups. By the descending chain condition the sequence must stabilize at a finite level, so that a minimal reduction must exist.

The uniqueness claimed in item 2 can be seen as follows. A G-invariant L_i-reduction, Q_i, yields a G-invariant H-equivariant map

$$\mathcal{G}_i : P \to H/L_i.$$

Taking the product $\mathcal{G}_1 \times \mathcal{G}_2$, we obtain a G-invariant, H-equivariant map

$$\mathcal{G} : P \to H/L_1 \times H/L_2.$$

The right-hand side is an H-space for the natural product action. Applying the measurable reduction lemma to $\mathcal{G}$, we conclude that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit in $H/L_1 \times H/L_2$, where U is a conull subset of M. We denote that orbit by $H \cdot (h_1 L_1, h_2 L_2)$. The isotropy group of $(h_1 L_1, h_2 L_2)$ is

$$L = \{h \in H \mid hh_1 L_1 = h_1 L_1, hh_2 L_2 = h_2 L_2\}$$

and we have a G-invariant measurable L-reduction Q of P. Notice that $L \subset h_1 L_1 h_1^{-1} \cap h_2 L_2 h_2^{-1}$. L cannot be a proper subgroup of $h_i L_i h_i^{-1}$ since, otherwise, $Q h_i$ would define a proper reduction of Q_i, contradicting the minimality of Q_i. Therefore, $Q h_i = Q_i$, $i = 1, 2$, proving 2. Notice that the same argument also shows 3. $\qquad\square$

We give next the C^r counterpart of the previous result.

Proposition 6.5 Let P be a principal H-bundle over a manifold M. Suppose that a group G acts by bundle automorphisms of P so that the action on M is topologically transitive. Then, for each $r \geq 0$, there exists a real algebraic subgroup $L \subset H$ and a G-invariant C^r L-reduction $Q \subset P|_U$, over a G-invariant dense open subset $U \subset M$, such that Q is minimal in the same sense already defined in the previous proposition. Moreover, the conclusions 2 and 3 of Proposition 6.4 also hold here after replacing "measurable" by "C^r" and taking into account that all reductions are only defined over a G-invariant open and dense subset of M.

Proof. This is shown following the same lines of the previous proof, using the C^r form of the reduction lemma. $\qquad\square$

The conjugacy class of the group L obtained above is called the C^r (resp., the measurable) *algebraic hull* of the G-action on P. By abuse of language, we sometimes call L itself the algebraic hull. If the action is not ergodic, we should regard the algebraic hull as a map from the ergodic components of a quasi-invariant measure into the conjugacy classes of algebraic subgroups of P.

If G is a Lie group that acts via a smooth action on a manifold M, then G also acts on each of the frame bundles $F^r(M)$. An interpretation of Propositions 6.4 and 6.5 is that it is possible to define a "maximal A-structure" of order r that is invariant under G and is, in a sense, unique.

The interest in understanding the algebraic hull of a G-action begins to be justified by the next proposition. It says, in part, that the algebraic hull of the action sets a "lower bound" on the size of the groups $\mathrm{Iso}_{x,x}^i(M,\mathcal{G})$ for any A-structure $\mathcal{G}$ invariant under the action and almost all $x \in M$.

Proposition 6.6 Let G be a Lie group that acts smoothly on a manifold M so as to preserve a geometric A-structure of order r. Let $L_i \subset G^{r+i}$ be (a representative) of the measurable algebraic hull of the G-action (by automorphisms on $F^{r+i}(M)$) with respect to an ergodic quasi-invariant measure on M. Then for almost all $x \in M$, and each $i \geq 0$, the group $\mathrm{Iso}_{x,x}^i(M,\mathcal{G})$ contains a subgroup isomorphic to L_i.

Proof. Recall that each $\mathcal{G}^i$ is an equivariant map from $F^{r+i}(M)$ into some algebraic variety V^i upon which G^{r+i} acts algebraically. By Proposition 6.2, over a set of full measure in M, $\mathcal{G}$ takes values into a single G^{r+i}-orbit $G^{r+i} \cdot v_0$ in V^i. Therefore, $\mathcal{G}^i$ can be described as an L_{v_0}-structure, where L_{v_0} is the isotropy subgroup of v_0 in G^{r+i}. Recall that an element of $\mathrm{Iso}_{x,x}^i(M,\mathcal{G})$ can be described as a pair (ξ,η) (modulo G^{r+i}), where ξ, η lie in the fiber of $F^{r+i}(M)$ above x and $\mathcal{G}^i(\xi) = \mathcal{G}^i(\eta)$. It follows that $\mathrm{Iso}_{x,x}^i(M,\mathcal{G})$ is isomorphic to L_{v_0}. On the other hand, the algebraic hull L_i is contained in L_{v_0}, by Proposition 6.12. $\qquad\square$

Corollary 6.7 Suppose that $\mathcal{G}$ is an analytic rigid A-structure. For i sufficiently large $\mathrm{Iso}_{x,x}^{\mathrm{loc}}(M,\mathcal{G})$ contains a group isomorphic to L_i for each $x \in M$. Furthermore, for each $x \in M$ and each x' in the universal covering $\tilde{M}$ in the fiber of x, the space of (globally defined) Killing vector fields on $\tilde{M}$ (of the lift of $\mathcal{G}$ to $\tilde{M}$) vanishing at x' contains a Lie algebra isomorphic to the Lie algebra of L_i.

Proof. The first assertion follows from Theorem 4.9(1). The fact about Killing fields is a consequence of 4.9(2). $\qquad\square$

We give next a few applications of the dynamical ideas discussed so far, for actions of semisimple Lie groups .

Lemma 6.8 Let M be a G-space with an ergodic G-invariant probability measure μ. Let $\rho : G \to GL(V)$ be a representation of G on the finite dimensional (real) vector space V. Denote by H the Zariski closure of $\rho(G)$ in $GL(V)$ and suppose that $\rho(G)$ is a subgroup of finite index in H. We assume moreover that H is generated by algebraic 1-parameter subgroups. Then H is the algebraic hull of the G-action by bundle automorphisms of the (trivial) principal H-bundle $P = M \times H$ given by $g(x,h) := (gx, \rho(g)h)$.

Proof. Let $L \subset H$ denote the algebraic hull and let Q be a G-invariant measurable L-reduction of P. The reduction is, in effect, a G-invariant measurable assignment of an L-orbit (for the right-translation L-action on H) at each $x \in M$, i.e., a G-invariant measurable section of the fiber bundle P/L, whose standard fiber is H/L. In the present situation, where P is already a product, having an L-reduction is equivalent to having a measurable map $\phi : M \to H/L$ such that for each $x \in M$, $g(x, \phi(x)) = (gx, \phi(gx))$. Therefore, ϕ has the property

$$\phi(gx) = \rho(g)\phi(x)$$

for all $g \in G$ and all $x \in M$. The probability measure $\phi_*\mu$ on H/L is $\rho(G)$-invariant since $\phi_*\mu = \phi_*g_*\mu = \rho(g)_*\phi_*\mu$, for each $g \in G$. By averaging $\phi_*\mu$ over the finite group $H/\rho(G)$ we obtain an H-invariant probability measure on H/L. We can now apply Corollary 10.5 (appendix C) to conclude that $H = L$. $\qquad\square$

The next corollary and lemma are from [24].

Corollary 6.9 Let G be a noncompact connected simple Lie group and let M be a G-space with an ergodic G-invariant probability measure μ. Denote by H the Zariski closure of $\mathrm{Ad}(G)$ in $GL(\mathfrak{g})$. Then H is the algebraic hull of the G-action by bundle automorphisms on the (trivial) principal H-bundle $P = M \times H$ by $g(x,h) := (gx, \mathrm{Ad}(g)h)$.

Lemma 6.10 Let G be a noncompact connected simple Lie group that acts on a connected manifold M with a finite G-invariant measure positive on open sets. Denote by G_x the isotropy subgroup of $x \in M$. Then, if the action is not trivial, there exists an open dense set of full measure on which G_x is discrete.

Proof. All that will be needed for the next theorem is that for each ergodic component of the measure either $G_x = G$ at almost every $x \in M$ or G_x

is discrete at almost every x. We first check this claim. Let $\mathfrak{g}_x$ be the Lie algebra of G_x and $\mathrm{Gr}(\mathfrak{g})$ the union of the Grassmann varieties of subspaces of $\mathfrak{g}$. Define $\phi : M \to \mathrm{Gr}(\mathfrak{g})$ by $\phi(x) = \mathfrak{g}_x$. Then ϕ is easily seen to be measurable and for each $g \in G$ and $x \in M$

$$\phi(gx) = \mathrm{Ad}(g)\phi(x).$$

If μ is the G-invariant probability measure on M, $\phi_*\mu$ is an $\mathrm{Ad}(G)$-invariant probability measure on $\mathrm{Gr}(\mathfrak{g})$. Since $\mathrm{Ad}(G)$ is a finite index subgroup of its Zariski closure H in $GL(\mathfrak{g})$, we obtain, as in the proof of the previous lemma, an H-invariant probability measure on $\mathrm{Gr}(\mathfrak{g})$. That invariant measure must be supported on the set of H-fixed points (Corollary 10.5, appendix C), so that over a set of full measure in M, the map ϕ takes values in the set of H-fixed points. In particular, there is a conull G-invariant subset $S \subset M$ such that $\phi|_S$ is a G-invariant function. By ergodicity, ϕ is constant almost everywhere. Call $\mathfrak{l}$ the constant value of ϕ. Then $\mathfrak{l} = \mathrm{Ad}(g)\mathfrak{l}$ for all $g \in G$, hence $\mathfrak{l}$ is an ideal of $\mathfrak{g}$. But the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and 0, so that G_x is either G or a discrete subgroup at almost every x.

We now drop the assumption that the measure is ergodic, and suppose that it is positive on open sets. Let $\Lambda \subset M$ be the (measurable) subset where $G_x = G$. We want so show that Λ has measure 0. Suppose for a contradiction that this is not the case and let K be the maximal compact subgroup of G. subgroup of G. The action of K can be linearized at each of its fixed points. (Choose a K-invariant Riemannian metric on M and consider a normal neighborhood near the point. Then in exponential coordinates the local action will be linear.) Therefore, each density point of Λ has a neighborhood where K acts trivially. It is clear that if x is in the closure of an open set where K is trivial, then x is fixed by K and has a neighborhood of fixed points. (Again by linearization.) Therefore, K acts trivially on all of M. On the other hand, the subgroup of G that acts trivially on M is a normal subgroup. Since G is simple we obtain the desired contradiction. $\square$

Theorem 6.11 (Zimmer) Let G be a connected, noncompact, simple Lie group, acting nontrivially on a compact n-dimensional manifold M. Suppose that the action preserves an H-structure on M where H is a real algebraic subgroup of $GL(n, \mathbb{R})$ consisting of matrices of determinant ± 1. Then there is a Lie algebra embedding $\pi : \mathfrak{g} \to \mathfrak{h}$ such that the representation π of $\mathfrak{g}$ on $\mathbb{R}^n$ contains $\mathrm{ad}(\mathfrak{g})$ as a subrepresentation.

Proof. We have pointed out before that if an action preserves an H-structure on M such that H consists of matrices of determinant 1, then it also pre-

serves a nonvanishing alternating n-form on M. Similarly, if we allow the elements of H to have determinant either 1 or -1, then the action preserves a nonvanishing n-form which is only well-defined up to sign, i.e. a *volume density*. This nevertheless allows us to define a smooth G-invariant measure on M. The total measure of M is finite since M is compact, so after normalization we may assume that M admits a G-invariant probability measure μ whose support is the entire M.

For each ergodic component of μ we can apply Lemma 6.8 and conclude that G_x is either discrete or equal to G for μ-a.e. $x \in M$. If $G_x = G$ for μ-a.e. x, we have by continuity that the action is trivial, contrary to the hypothesis. Therefore, there must be a G-invariant measurable subset $S \subset M$ of positive μ-measure such that G_x is discrete for all $x \in S$.

At each $x \in S$, the differential of the orbit map $\tau_x : G \to M$, $\tau_x(g) := gx$, yields an identification of the tangent space at x of the G-orbit of x with the Lie algebra of G, as indicated by the following arrow.

$$F_x := (D\tau_x)_e : \mathfrak{g} \xrightarrow{\cong} V_x := T_x(G \cdot x)$$

Moreover, with respect to this identification, the derivative action of each $g \in G$ on the G-invariant subbundle V of TM with fibers V_x is given by $\mathrm{Ad}(g)$, i.e. $Dg_x : V_x \to V_{gx}$ and

$$F_{gx} \circ \mathrm{Ad}(g) = Dg_x \circ F_x.$$

Let m be the dimension of $\mathfrak{g}$ and view $\mathfrak{g}$ as the subspace $\mathbb{R}^m \subset \mathbb{R}^n$, corresponding to setting equal to 0 the last $n - m$ coordinates of $\mathbb{R}^n$. Let H_1 denote the image of G in $GL(m, \mathbb{R})$ under the adjoint representation. Then by the above discussion, we obtain over S a measurable $\overline{H}$-reduction of the frame bundle $F(M)|_S$ where $\overline{H}_1$ is a subgroup of $GL(n, m, \mathbb{R})$ that restricts to H on the invariant subspace $\mathbb{R}^n$. We can now apply Corollary 6.9 to conclude that the algebraic hull of the action contains $\mathrm{Ad}(G)$. But some conjugate of the algebraic hull is contained in H. Since the Lie algebra of $\mathrm{Ad}(G)$ is isomorphic to $\mathfrak{g}$, we obtain that some conjugate of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{h}$. $\qquad\square$

It can also be shown (see [24]) that if $\mathfrak{g}$ is the Lie algebra of a noncompact simple Lie group and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a Lie algebra homomorphism such that on V there is a nondegenerate symmetric bilinear form of signature $(1, \dim V - 1)$, invariant under $\pi(\mathfrak{g})$, and $\mathrm{ad}(\mathfrak{g})$ is a subrepresentation of π, then $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Therefore, by the previous theorem, if a connected noncompact simple Lie group G acts nontrivially on a compact manifold

preserving a Lorentz metric (i.e., a pseudo-Riemannian metric of signature $(1, \dim V - 1)$), then G is locally isomorphic to $SL(2, \mathbb{R})$.

We say that a geometric structure is *unimodular* if it determines an L-reduction of $F^1(M)$ where L is the group of matrices with determinant 1 or -1. In other words, the structure is unimodular if it incorporates a volume density.

Proposition 6.12 Suppose that G is a noncompact simple Lie group with finite center acting nontrivially on a compact manifold M so as to preserve a unimodular A-structure $\mathcal{G}$. Then, for almost all $x \in M$ and each $i \geq 1$, the algebraic hull L_i (and, therefore, also $\mathrm{Iso}^i_{x,x}(M, \mathcal{G})$) contains a group locally isomorphic to G.

Proof. This is a consequence of (the proof of) Theorem 6.11 together with Corollary 6.7. $\qquad\square$

Corollary 6.13 For i sufficiently large and almost all $x \in M$, $\mathrm{Iso}^{\mathrm{loc}}_{x,x}(M, \mathcal{G})$ contains a group locally isomorphic to G.

Proof. This is due to the proposition and Theorem 4.9(1). $\qquad\square$

7 Rigid structures and the topology of M

In all of the section, M will be a compact real analytic manifold, equipped with a real analytic, rigid, unimodular A-structure $\mathcal{G}$. G will denote a connected noncompact simple Lie group with finite center that acts analytically by isometries of $\mathcal{G}$. The action in assumed to be nontrivial.

Theorem 7.1 (Gromov) Under the assumptions of the previous paragraph, there exists an integer m and a representation $\rho : \pi_1(M) \to GL(m, \mathbb{R})$ such that the Zariski closure of the image of ρ contains a group locally isomorphic to G.

In particular, $SL(2, \mathbb{R})$, for example, cannot act nontrivially on a sphere S^n so as to preserve the volume form and an analytic connection. More generally,

Corollary 7.2 G cannot act analytically and nontrivially on a compact manifold M with amenable fundamental group leaving invariant an analytic rigid A-structure.

Proof. This is due to the fact that the Zariski closure of an amenable group is amenable. See Theorem 4.1.15 in [22]. $\qquad\square$

The above theorem, used in combination with Zimmer's cocycle super-rigidity theorem and Ratner's solution of Raghunathan's conjecture was used by Zimmer and Zimmer-Lubotzky [17] to obtain detailed information about the fundamental groups of manifolds supporting actions by (higher rank) semisimple Lie groups. As an illustration, we state without proof (part of) a theorem given in [25]. The reader will find the complete statement and further applications also in [23].

Theorem 7.3 (Zimmer) Assume that G has finite fundamental group, its $\mathbb{R}$-rank is at least 2, and $\pi_1(M)$ admits a faithful linear representation $\sigma : \pi_1(M) \to GL(q, \mathbb{C})$, for some q, such that $\sigma(\pi_1(M))$ is discrete. Then, if G acts on M as in Theorem 7.1, $\pi_1(M)$ contains a lattice in a linear Lie group L, where L contains a group locally isomorphic to G.

The remainder of the section is dedicated to proving Theorem 7.1.

Let $\mathfrak{g}$ be the Lie algebra of G. Since G acts isometrically on M, $\mathfrak{g}$ can be identified with a Lie algebra of Killing fields on the universal covering $\tilde{M}$. Let s be large enough so that the conclusion of Theorem 4.3 holds. For almost all $x \in M$, let $L = L(x)$ be the algebraic hull at x of the action on $F^{r+s}(M)$. Let $x' \in \tilde{M}$ be in the fiber of x and denote by $\mathfrak{l}$ the Lie algebra of (global) Killing fields on $\tilde{M}$ that vanish at x' and is isomorphic to the Lie algebra of L. (Cf. Corollary 6.7.)

Lemma 7.4 The Lie algebra $\mathfrak{l}$ normalizes $\mathfrak{g}$, so that the Lie algebra homomorphism $X \mapsto [X, \cdot]$ from $\mathfrak{l}$ into the algebra of derivations of $\mathfrak{g}$ is onto $\mathrm{ad}(\mathfrak{g})$.

Proof. Let $J^{r+s}TM$ denote the vector bundle of $r + s$-jets of local vector fields on M. Denote by $\bar{\mathfrak{g}}(x)$ the subspace of the fiber of $J^{r+s}TM$ at x consisting of the $k + s$-jets of Killing fields induced from elements of $\mathfrak{g}$ by the action. According to Lemma 6.8, $\bar{\mathfrak{g}}(x)$ is isomorphic to $\mathfrak{g}$ for almost all x. Call the natural isomorphism $\varphi_x : \mathfrak{g} \to \bar{\mathfrak{g}}(x)$. It is clear that $g\bar{\mathfrak{g}}(x) = \bar{\mathfrak{g}}(gx)$ for each $g \in G$ and almost all x and that

$$g\varphi_x(X) = \varphi_{gx}(\mathrm{Ad}(g)X).$$

($\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation of G.) In particular, by the general properties of the algebraic hull, L may be chosen so that it also

44

preserves the (measurable) subbundle $x \mapsto \mathfrak{g}(x)$. Arguing as in the proof of Theorem 6.11 (see also Corollary 6.9) we obtain that the connected component L^0 of L acts on $\bar{\mathfrak{g}}(x) \cong \mathfrak{g}$ by $\mathrm{Ad}(G)$. Since Killing fields are uniquely determined by their $r+s$-jets, $\mathfrak{g}$ is normalized by L^0, hence also by the Lie algebra $\mathfrak{l}$. Furthermore $\mathfrak{l}$ acts on $\mathfrak{g}$ by the adjoint representation. $\qquad \square$

Corollary 7.5 Let $\mathfrak{g}$ be the Lie algebra of G, viewed as the algebra of Killing fields on $\tilde{M}$. Let $\mathfrak{z}$ be the centralizer of $\mathfrak{g}$ in the Lie algebra of all Killing fields on $\tilde{M}$. For each $x \in \tilde{M}$, let $\mathfrak{z}(x)$ and $\mathfrak{g}(x)$ be the respective images under the evaluation map at x. Then, $\mathfrak{g}(x)$ is contained in $\mathfrak{z}(x)$ for almost all x.

Proof. Let $x \in \tilde{M}$ be any point for which Lemma 6.10 holds. If $g \in G$ is sufficiently close to the identity, it is possible to choose $u \in L = L(x)$, also close to the identity, such that u acts on $\mathfrak{g}$ by $\mathrm{Ad}(g)^{-1}$. The composition $g \circ u$ is a local diffeomorphism near x that acts trivially on $\mathfrak{g}$ (recall that we can identify $\bar{\mathfrak{g}}(x)$ with $\mathfrak{g}$ as in the proof of Lemma 6.10) and sends x to gx. Let Z be the local group near x with Lie algebra $\mathfrak{z}$. Then, we have just shown that in the G-orbit of x there is an open neighborhood of x that also lies in the local Z-orbit of x. It follows that $\mathfrak{g}(x) \subset \mathfrak{z}(x)$, as claimed. $\qquad \square$

The following remark may help put the last result in perspective. Suppose that G is a subgroup of a simply connected Lie group H, acting on $M = H/\Gamma$ by left translations, where Γ is a discrete subgroup of H. On the universal covering H of M there is also a right action of G, which commutes with the left action. The corresponding vector fields are the elements of $\mathfrak{z}$ constructed above. The geometric structure in this case can be taken to be an invariant affine connection on H/Γ.

We now conclude the proof of Theorem 7.1. There is no loss of generality in supposing that that G is simply connected, so that we can lift the G-action on M to a global action of G on $\tilde{M}$. Let $\mathfrak{z}$ be as in Corollary 7.5. Notice that $\mathfrak{z}$ is invariant under $\pi_1(M)$. In fact, if $\gamma \in \pi_1(M)$, $X \in \mathfrak{z}$ and $Y \in \mathfrak{g}$, then $\gamma_* Y = Y$ (since Y is the lift of a vector field on M) and

$$[\gamma_* X, Y] = [\gamma_* X, \gamma_* Y] = \gamma_* [X, Y] = 0$$

so that $\gamma_* X$ is also in $\mathfrak{z}$. Therefore, we obtain a representation

$$\eta : \pi_1(M) \to GL(\mathfrak{z}).$$

Form the associated vector bundle

$$E := \tilde{M} \times_{\eta} \mathfrak{z} := (\tilde{M} \times \mathfrak{z})/\pi_1(M) \to M$$

and let G act on E by $g \cdot (x, Z)\pi_1(M) = (gx, Z)\pi_1(M)$.

The evaluation map $\mathrm{v} : \tilde{M} \times \mathfrak{z} \to T\tilde{M}$ is $\pi_1(M)$-equivariant, that is,

$$\mathrm{ev}(\gamma(x), X(x)) = d\gamma_x X(x),$$

and $T\tilde{M}/\pi_1(M) = TM$. So we obtain a homomorphism of vector bundles $\overline{\mathrm{ev}} : E \to TM$. The fact that $X \in \mathfrak{z}$ implies that $dg_x X(x) = X(gx)$ (on $\tilde{M}$), from which it follows that $\overline{\mathrm{ev}}$ commutes with the G-actions on E and TM. By Corollary 7.5, the image of $\overline{\mathrm{ev}}$ contains the measurable subbundle $\mathfrak{g}(x)$ of tangent spaces to the G-orbits.

Since the algebraic hull for the action of G on the frame bundle associated to the vector bundle $x \mapsto \mathfrak{g}_x$ contains $\mathrm{Ad}(G)$, it follows that the algebraic hull for the G-action on the $GL(\mathfrak{z})$-principal bundle

$$P := (\tilde{M} \times GL(\mathfrak{z}))/\pi_1(M))$$

also contains $\mathrm{Ad}(G)$. On the other hand, the algebraic hull of the G-action on P is contained in the Zariski closure, H, of the image of $\pi_1(M)$ under the representation η. In fact, P contains a G-invariant H-reduction, given by $(\tilde{M} \times GL(\mathfrak{z}))/\pi_1(M)$. With this remark, the proof of Theorem 7.1 is complete.

8   Appendix A - Basic concepts in dynamics

G-**spaces.** Let G be a group and X a set. A G-*action* on X is a map $\Phi : G \times X \to X$ that satisfies the following two properties:

1. $\Phi(e, x) = x$ for all $x \in X$, where e is the identity of G.

2. $\Phi(g_2, \Phi(g_1, x)) = \Phi(g_2 g_1, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

For each $g \in G$, let $\Phi_g : X \to X$ be defined by $\Phi_g(x) := \Phi(g, x)$. Then Φ_g is a bijection from X onto itself, with inverse $\Phi_{g^{-1}}$, and the map $g \mapsto \Phi_g$ from G into the group of bijective self-maps of X is a group homomorphism. We often write $g \cdot x$ or $g(x)$, or simply gx, instead of $\Phi(g, x)$. The definition of G-action just given is usually called a *left-action* of G. By a *right-action* of G on M we mean a map $\Phi : M \times G \to M$ such that 2 above is replaced with

$$\Phi(\Phi(x, g_1), g_2) = \Phi(x, g_1 g_2).$$

For each $x \in X$, we define the *orbit* of x by

$$Gx := \{\Phi_g(x) \mid g \in G\}.$$

The orbits of a G-action partition X into disjoint sets, namely Gx are the equivalence classes of the relation

$x \sim y$ if and only if there exists $g \in G$ such that $x = gy$.

The *orbit space* is the set of equivalence classes, denoted $G\backslash X$. The action Φ is called *transitive* if the G-space has only one orbit, i.e. $X = Gx$ for some x.

Typically, the G-action will leave invariant, or preserve, some structure on X such as a topology, a measurable structure, a smooth manifold structure, or a structure of algebraic variety.

Let H be a closed subgroup of G. Then the coset space

$$G/H = \{gH \mid g \in G\}$$

has the quotient topology induced by the natural projection $\pi : G \to G/H$, $\pi(g) = gH$, namely, the open subsets of G/H are $\pi(U) = \{gH \mid g \in U\}$ for all open sets $U \subset G$. With respect to the quotient topology, π is continuous and open and G/H is a Hausdorff space.

The *kernel* of an action Φ, denoted $\mathrm{Ker}(\Phi)$, is the kernel of the homomorphism $g \mapsto \Phi_g$, which is a normal subgroup of G. When $\mathrm{Ker}(\Phi)$ is trivial, the action is said to be *effective*. If the action is not effective, Φ induces an effective action of $G/\mathrm{Ker}(\Phi)$ on X. The action is called *locally effective* if $\mathrm{Ker}(\Phi)$ is a discrete subgroup of G.

For each $x \in X$, the *isotropy group of x* is defined by

$$G_x := \{g \in G \mid gx = x\}.$$

G_x is a subgroup of G and it is immediate that $G_{gx} = gG_xg^{-1}$ for each $g \in G$ and $x \in X$. Moreover, $\mathrm{Ker}(\Phi) = \bigcap_{x \in X} G_x$. If $G_x = \{e\}$ for all $x \in G$, we say that the G-action is *free*. The action is called *locally free* if G_x is a discrete subgroup of G for all x in X.

A topological space X is said to be T_1 if each point $x \in X$ is closed. It is an easy consequence of the definitions that whenever X is a T_1 G-space, each isotropy group G_x as well as the kernel of Φ are closed subgroups of G, and that $G/\mathrm{Ker}(\Phi)$ is a topological group in a natural way. Moreover, the induced (effective) action of $G/\mathrm{Ker}(\Phi)$ makes X a topological $G/\mathrm{Ker}(\Phi)$-space.

The Orbit Space. Until we make any further requirement G will be a locally compact second countable topological group and X a complete second countable metrizable G-space. We give $G\backslash X$ the quotient topology induced by the natural projection that to each $x \in X$ associates its orbit.

We say that the σ-algebra $\mathcal{B}$ of Borel sets, i.e. the σ-algebra generated by the open sets in $G\backslash X$, is *countably separating* if there is a sequence $B_i \in \mathcal{B}$ such that for each pair of points in X one can find a B_i that contains exactly one of the two points. In this case, the G-action will be called *tame*.

The orbit Gx of a topological G-space X is *locally closed* if it is open in its closure $\overline{Gx} \subset X$. The next theorem due to Glimm and Effros gives a useful characterization of tame actions. The proof can be found in [22].

Theorem 8.1 Suppose that Φ is a continuous action of a locally compact second countable group G on a complete second countable metrizable space X. Then the following are equivalent:

1. All orbits are locally closed.

2. The action is tame.

3. For every $x \in X$, the natural map $G/G_x \to Gx$ is a homeomorphism, where Gx has the relative topology as a subset of X.

An element x in a G-space X is said to be a *fixed point* if $G_x = G$. It is a *periodic point* if G/G_x is compact. A (topological) G-space X is said to be *topologically transitive* if some G-orbit is dense in X. If *all* orbits are dense, the action is called *minimal*. A subset $A \subset X$ is called *G-invariant* if for each $x \in A$ and $g \in G$, $gx \in A$. An equivalent definition of minimal action is that X does not have a proper closed G-invariant set, since the closure of a G-invariant set is a G-invariant set. A point x of a topological G-space X will be called *recurrent* if for each neighborhood U of x and each compact $K \subset G$, there is g in the complement of K such that $gx \in U$. It is immediate from the definitions that periodic points are recurrent. Furthermore, if both the orbit of x and its complement are dense in X, then x is a recurrent point. We leave the verification of this last claim as an exercise to the reader. Notice that the action of G on itself by translations is topologically transitive—in fact transitive—but not recurrent.

Invariant measures. Let X be a measurable space with σ-algebra $\mathcal{A}$ and T a measurable map from X into to another measure space Y with σ-algebra $\mathcal{B}$. If μ is a measure on $(X, \mathcal{A})$, we define $T_*\mu$ as the measure on $(Y, \mathcal{B})$ such that
$$T_*\mu(B) := \mu(T^{-1}B)$$

for each $B \in \mathcal{B}$.

A measurable map T of a measure space $(X, \mathcal{A}, \mu)$ into itself is said to be *measure preserving* if $T_*\mu = \mu$. We also say that μ is an *invariant measure* for T. If $T_*\mu$ and μ are in the same measure class (i.e., if they have the same sets of measure 0) we say that μ is a *quasi-invariant* measure for T. We say that T is a *measure preserving transformation* of $(X, \mathcal{A}, \mu)$ if it is bijective and T and T^{-1} are measurable and measure preserving.

Let G be a locally compact second countable topological group. Let $(X, \mathcal{A}, \mu)$ be a measure space. A *measure preserving action* of G on X is an action defined by a measurable map $\Phi : G \times X \to X$, where $G \times X$ has the product measurable structure, and the maps $\Phi_g := \Phi(g, \cdot)$, $g \in G$, are measure preserving transformations of X. Of course, a G-space defined by a continuous G-action is also a measurable G-space for the σ-algebra of Borel sets. We say that X is a *G-space with invariant measure μ*.

One may also consider a G-space with *quasi-invariant measure μ*, in which case the G-action preserves only the measure class of μ. As before, we will often write $\Phi_g(x)$ simply as gx. In this notation, a G-invariant measure μ satisfies $g_*\mu = \mu$.

If a G-space admits a measure μ that is both invariant and finite, then the $\mathbb{Z}$-action generated by each element of G has following recurrence property.

Theorem 8.2 (Poincaré Recurrence) Let X be a $\mathbb{Z}$-space, the $\mathbb{Z}$-action being generated by a transformation g that preserves a probability measure μ on X. Then for any set $E \in \mathcal{A}$, μ-almost every point $x \in E$ returns infinitely often to E. More precisely, there is a measurable subset F of E such that $\mu(F) = \mu(E)$ and, for each $x \in F$, a sequence $n_1 < n_2 < \ldots$ such that $g^{n_i} x \in E$ for all i.

Ergodicity. A measurable map $F : X \to Y$ between G-spaces is called a *G-map* if

$$F(gx) = gF(x)$$

for all $g \in G$ and $x \in X$. If the G-action on $F(X)$ is trivial, we say that F is *G-invariant*. Given a quasi-invariant measure μ on X, we say that a measurable map F between G-spaces X and Y is a *G-map relative to μ* if for each $g \in G$

$$\mu(\{x \in X : F(gx) \neq gF(x)\}) = 0.$$

A G-space X, with a quasi-invariant measure μ, is said to be *ergodic* if every G-invariant measurable set is either null (i.e, it has zero measure) or conull (i.e., its complement has zero measure). We also say that μ is an

ergodic measure for the G-space X. Therefore, the action is ergodic if X cannot be decomposed as the disjoint union of two G-invariant measurable subsets, both with positive measure.

A nonergodic G-invariant measure can be "disintegrated" in terms of its "ergodic components." The following theorem makes this precise.

Theorem 8.3 (Ergodic decomposition) Let X be a compact metrizable space with a continuous action of a locally compact second countable group G, and let μ be a finite Borel measure on X. Then, there exists a measure space $\mathcal{E}$ with measure ν and for each $\alpha \in \mathcal{E}$ a measure space $(X_\alpha, \mathcal{A}_\alpha, \mu_\alpha)$ such that the sets $\{X_\alpha\}_{\alpha \in \mathcal{E}}$ form a partition of X into measurable G-invariant subsets and

1. for any measurable set $A \subset X$, $A \cap X_\alpha$ belongs to $\mathcal{A}_\alpha$ for ν-almost every $\alpha \in \mathcal{E}$ and

$$\mu(A) = \int_{\mathcal{E}} \mu_\alpha(A \cap X_\alpha) \, d\nu(\alpha)$$

2. for ν-almost every $\alpha \in \mathcal{E}$, X_α is an ergodic G-space relative to the measure μ_α.

The following proposition shows one way in which ergodicity relates to topological properties of the action.

Proposition 8.4 Suppose that X is a second countable topological space and that one is given a continuous action of a locally compact second countable topological group G. Suppose that the action is ergodic relative to a quasi-invariant measure μ which is positive on open sets. Then for almost every $x \in X$, the orbit $\{gx : g \in G\}$ is dense in X.

A measurable space is called *countably separated* if there is a countable family of measurable sets that separates points.

Proposition 8.5 Suppose that X is an ergodic G-space, Y is a countably separated measurable space and $f : X \to Y$ is a strict G-invariant measurable function. Then f is almost everywhere constant.

9 Appendix B - Semisimple Lie groups

We only consider here linear Lie groups, that is, (real) subgroups of $GL(n, \mathbb{C})$. We denote by A^* the complex conjugate transpose of A. The *Cartan involution* of $GL(n, \mathbb{C})$ is the homomorphism

$$\Theta : A \mapsto (A^*)^{-1}.$$

The Cartan involution of $\mathfrak{gl}(n, \mathbb{C})$ (the Lie algebra of $GL(n, \mathbb{C})$) is the Lie algebra isomorphism induced from Θ, and is given by $\theta : X \mapsto -X^*$. Θ is indeed a group homomorphism and an involution, i.e. Θ^2 is the identity map, as one can easily check.

Let G be a connected Lie subgroup of $GL(n, \mathbb{C})$. We say that G is a *reductive* group if it is conjugate to a subgroup that is stable under the Cartan involution Θ. In other words, G is reductive if there is $g \in GL(n, \mathbb{C})$ such that gGg^{-1} is mapped into itself by Θ. A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ is reductive if it is conjugate by an element in $GL(n, \mathbb{C})$ to a θ-stable subalgebra. In particular, G is reductive if and only if $\mathfrak{g}$ is.

The center of a group G is the subgroup

$$Z(G) = \{a \in G \mid ag = ga \text{ for all } g \in G\}.$$

It is clearly a normal subgroup of G. The center of a Lie algebra $\mathfrak{g}$ is the subalgebra

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$$

and is an ideal of $\mathfrak{g}$. (A subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is an ideal if $[X, Y] \in \mathfrak{n}$ for all $X \in \mathfrak{n}$ and all $Y \in \mathfrak{g}$.)

A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ is *semisimple* if it is reductive and has trivial center. $G \subset GL(n, \mathbb{C})$ is a *semisimple Lie group* if its Lie algebra is semisimple.

A Lie algebra is said to be *simple* if its only ideals are $\{0\}$ and itself.

Given next is a list of the classical semisimple groups. (Most are, in fact, simple.) I_n is the identity matrix of size n and

$$J_{2n} := \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad K_{p,q} := \begin{pmatrix} I_{p,q} & 0 \\ 0 & I_{p,q} \end{pmatrix}, \quad I_{p,q} := \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

$$
\begin{aligned}
SL(n,\mathbb{C}) &= \{A \in GL(n,\mathbb{C}) \mid \det A = 1\} \\
SL(n,\mathbb{R}) &= \{A \in GL(n,\mathbb{R}) \mid \det A = 1\} \\
Sp(2n,\mathbb{C}) &= \{A \in GL(2n,\mathbb{C}) \mid A^t J_{2n} A = J_{2n}\} \\
Sp(2n,\mathbb{R}) &= \{A \in GL(2n,\mathbb{R}) \mid A^t J_{2n} A = J_{2n}\} \\
SO^*(2n) &= \{A \in SL(2n,\mathbb{C}) \mid A^t A = I_{2n},\ A^* J_{2n} A = J_{2n}\} \\
SU^*(2n) &= \{A \in SL(2n,\mathbb{C}) \mid A J_{2n} = J_{2n}\overline{A}\} \\
SO_n(\mathbb{C}) &= \{A \in SL(n,\mathbb{C}) \mid A^t A = I_n\} \\
SO(p,q) &= \{A \in SL(n,\mathbb{R}) \mid A^t I_{p,q} A = I_{p,q}\} \\
SU(p,q) &= \{A \in SL(n,\mathbb{C}) \mid A^* I_{p,q} A = I_{p,q}\} \\
Sp(p,q) &= \{A \in GL(2(p+q),\mathbb{C}) \mid \\
&\qquad A^* K_{p,q} A = K_{p,q},\ A^t J_{2(p+q)} A = J_{2(p+q)}\}.
\end{aligned}
$$

Real rank. Since θ is an involution, i.e. $\theta^2 = \mathrm{id}$, its only eigenvalues are 1 and -1. We define subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of the θ-stable Lie algebra $\mathfrak{g}$ as follows:

$$
\begin{aligned}
\mathfrak{k} &:= \{X \in \mathfrak{g} \mid \theta(X) = X\} \\
\mathfrak{p} &:= \{X \in \mathfrak{g} \mid \theta(X) = -X\}.
\end{aligned}
$$

Since θ is a Lie algebra automorphism, $\mathfrak{k}$ is a Lie subalgebra, but $\mathfrak{p}$ is only a subspace. As a vector space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Introduce in $\mathfrak{g}$ an inner product by

$$
\langle X, Y \rangle = -\mathrm{Re}(\mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(\theta Y)))
$$

where $\mathrm{ad}(X)$ is the linear map on $\mathfrak{g}$ defined by $\mathrm{ad}(X)Z = [X, Z]$. For each $X \in \mathfrak{p}$, the operator $\mathrm{ad}(X)$ on $\mathfrak{g}$ is self-adjoint with respect to the given inner product. Therefore $\mathrm{ad}(X)$ is diagonalizable with real eigenvalues. Let $\mathfrak{a}$ be a maximal abelian algebra in $\mathfrak{p}$. More precisely, $\mathfrak{a}$ is abelian and is not properly contained in a subspace of $\mathfrak{p}$ consisting of commuting elements. The operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$, commute since

$$
0 = \mathrm{ad}([X,Y]) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)
$$

for $X, Y \in \mathfrak{a}$. Therefore, it is possible to find a basis for $\mathfrak{g}$ which simultaneously diagonalizes all the operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$.

The subalgebra $\mathfrak{a}$ will be called an $\mathbb{R}$-*split Cartan subalgebra* of $\mathfrak{g}$. A more descriptive name is "maximal abelian $\mathbb{R}$-diagonalizable subalgebra." The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$.

10   Appendix C - Algebraic actions

Let G be a connected Lie group and let $\mathfrak{g}$ be its Lie algebra. Then G has a canonical representation into the group of real automorphisms of $\mathfrak{g}$, the *adjoint representation*, defined as

$$\mathrm{Ad} : G \to GL(\mathfrak{g})$$

such that $\mathrm{Ad}(g)(X) = gXg^{-1}$. Although the adjoint representation is not in general faithful, its kernel is precisely the center of G, as one can easily check. If G is semisimple, its center Z is a closed subgroup with trivial Lie algebra, therefore Z is discrete. Therefore, for a connected semisimple G, G/Z is isomorphic to $\mathrm{Ad}(G)$.

We recall that a *linear algebraic group* is a group of matrices defined by polynomial conditions on their entries.

Proposition 10.1 Let G be a connected semisimple Lie group and Z its center. Then the adjoint representation defines an isomorphism between G/Z and the connected component of the identity of the group of real points of a linear algebraic group defined over $\mathbb{R}$.

The algebraic group of the previous theorem is called the *adjoint group* of G. In particular, if G is a connected semisimple Lie group with trivial center, then G is naturally isomorphic to the identity component of a real algebraic group.

We state now a stratification theorem for algebraic actions due to Rosenlicht. (For general properties about algebraic groups and actions we refer the reader to [22] and Rosenlicht's own paper [19]. See also [6].)

Let V be an affine, projective, or more generally, a quasi-projective smooth real variety and let H be a linear real algebraic group. An *algebraic action* of H on V is a morphism

$$\Phi : H \times V \to V$$

such that Φ is a group action on V.

Theorem 10.2 (Rosenlicht) Let H, V, and Φ be as in the previous paragraph. There exists a finite partition of V into smooth, locally closed, H-invariant algebraic varieties

$$V = V_0 \cup V_1 \cup \cdots \cup V_l$$

such that, for each $0 \leq i \leq l$, the union $V_i \cup \cdots \cup V_l$ is Zariski closed in V and contains V_i as a Zariski-open subset. Moreover, for each i, V_i/H has the structure of a smooth real variety (in particular, a smooth manifold) and $V_i \to V_i/H$ is a smooth fibration.

An immediate consequence of the theorem is that the H-orbits are locally closed and embedded in V. This, together with theorem 8.1 implies the next fact.

Corollary 10.3 Let $\Phi : H \times V \to V$ be a real algebraic action. Then each orbit of Φ is locally closed and is an embedded submanifold of V. In particular, real algebraic actions are tame.

By a *real algebraic* 1-*parameter group* we mean a real algebraic group isomorphic to either $GL(1, \mathbb{R}) \cong \mathbb{R}^*$ or the additive group $\mathbb{R}$.

Corollary 10.4 Let $\Phi : H \times V \to V$ be a real algebraic action, where H is a 1-parameter real algebraic subgroup. Then any recurrent point is a fixed point.

Proof. Since orbits are embedded, any recurrent point $x \in V$ must actually be a periodic point. In that case, the isotropy group of x is the set of real points of a Zariski closed infinite algebraic subgroup of a 1-dimensional group, therefore $H_x = H$ and x is a fixed point. $\square$

Given an algebraic group H, let H' be the smallest algebraic subgroup of H containing all the real 1-parameter subgroups—a normal subgroup. We say that H is *generated by real algebraic* 1-*parameter subgroups* if $H' = H$. For example, a semisimple group without compact factors has this property.

Corollary 10.5 Let $\Phi : H \times V \to V$ be a real algebraic action and suppose that H is generated by real algebraic 1-parameter subgroups. Let μ be an H-invariant probability measure on V. Then μ is supported on the set of H-fixed points.

Proof. This is an immediate consequence of the previous corollary and Poincaré recurrence. $\square$

References

[1] A. M. Amores. *Vector fields of a finite type G-structure*, J. Differential Geometry 14 (1979) 1-6.

[2] Y. Benoist. *Orbites des structures rigides (d'après Gromov)*, Notes of lectures given at the Banach Center, Warsaw, 1995.

[3] Y. Benoist, P. Foulon, F. Labourie. *Flots d'Anosov à distribuition stable et instable différentiables*, J. Am. Math. Soc. **5** 33-74 (1992).

[4] Y. Benoist et F. Labourie. *Sur les difféomorphismes d'Anosov affines à feuilletages stable et instable différentiables*, Invent. math. 111, 285-308 (1993).

[5] M. Gromov and G. D'Ambra. *Lectures on transformation groups: Geometry and Dynamics*, J. Diff. Geom.

[6] R. Feres. Dynamical Systems and Semisimple Groups, Cambridge U. Press, 1998.

[7] Y. Gliklikh. Global Analysis in Mathematical Physics - Geometric and Stochastic Methods, in *Applied Mathematical Sciences* vol. 122, Springer-Verlag, 1997.

[8] H. Goldschmidt and S. Sternberg. *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier, Grenoble 23.1 (1973) 203-267.

[9] H. Goldschmidt. *Integrability criteria for systems of nonlinear partial differential equations*, J. Differential Geometry, 1 (1967) 269-307.

[10] M. Gromov. *Rigid transformation groups*, in Géométrie différentielle, Travaux en cours, 33, Hermann, Paris 1988.

[11] M. Gromov. Partial Differential Relations, Springer-Verlag, 1980.

[12] V. Lychagin. *Homogeneous Geometric Structures and Homogeneous Differential Equations*, Amer. Math. Soc. Transl. (2) Vol. 167, 1995.

[13] A. Katok and B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1995.

[14] S. Kobayashi. Transformation Groups in Differential Geometry, Springer-Verlag, 1972.

[15] D. Krupka. *Local Invariants of a Linear Connection*, in Colloquia Mathematica Societatis János Bolyai, 31. Differential Geometry, Budapest, 1979.

[16] I. Kolar, P. W. Michor, and Jan Slovak. *Natural Operations in Differential Geometry*, Springer-Verlag 1991.

[17] A. Lubotzky and R. Zimmer. *Arithmetic structure of fundamental groups and actions of semisimple Lie groups*, to appear in Topology.

[18] K. Mackenzie. *Lie Groupoids and Lie Algebroids in Differential Geometry*, Cambridge University Press, 1989.

[19] M. Rosenlicht. *A Remark on Quotient Spaces*, Anais da Academia Brasileira de Ciencias 35, no. 4 (1963):487-9.

[20] I. M. Singer. *Infinitesimally Homogeneous Spaces*, Comm. Pure and App. Math., Vol. XIII (1960) 685-697.

[21] V. S. Varadarajan. Lie Groups, Lie Algebras, and Their Representation Theory, Springer-Verlag, 1984.

[22] R. Zimmer. Ergodic theory and semisimple groups, Monographs in Mathematics, Birkhäuser, 1984.

[23] R. Zimmer. *Automorphism Groups and Fundamental Groups of Geometric Structures*, Proceedings of Symposia in Pure Mathematics, vol. 54 (1993), Part 3.

[24] R. Zimmer. *Ergodic Theory and the Automorphism Group of a G-structure*, In *Group Representations, Ergodic Theory, Operator Algebras, and Mathematical Physics*. (Ed. C. C. Moore) pp. 247-278, Springer, New York, 1987.

[25] R. Zimmer. *Superrigidity, Ratner's theorem, and fundamental groups*, Israel J. Math. 74 (1991) 199-207.