Math 132 - Exam I - Spring 2008

This exam contains 15 multiple choice questions and 2 hand graded questions. The multiple choice questions are worth 5 points each and the hand graded questions are worth a total of 25 points. The latter questions will be evaluated not only for having the correct solutions but also for clarity. Points may be taken for confusing and disorganized writing, even when the answer is correct.

1. Evaluate the sum

$$\sum_{j=2}^{30}(2j+j^2).$$

A) 10374

B) 10375

C) 10376

D) 10377

E) 10378

F) 10379

G) 10380

H) 10381

→ **I**) 10382

J) 10383

$$\begin{aligned}\sum_{j=2}^{30}(2j+j^2) &= -3+\sum_{j=1}^{30}(2j+j^2)\\ &= -3+2\sum_{j=1}^{30}j+\sum_{j=1}^{30}j^2\\ &= -3+2\left(\frac{30\times 31}{2}\right)+\frac{30\times 31\times 61}{6}\\ &= -3+930+9455\\ &= 10382.\end{aligned}$$

2. Find a formula for the Riemann sum R_N for the function $f(x) = x^2$ over the interval $[1, 2]$.

A) $R_N = \frac{1}{N} \sum_{j=0}^{N-1} (2 + \frac{j+1}{N})^2$

B) $R_N = \frac{1}{N} \sum_{j=1}^{N} (2 + \frac{j+1}{N})^2$

C) $R_N = \frac{2}{N} \sum_{j=1}^{N} (1 + \frac{j+1}{N})^2$

D) $R_N = \frac{1}{N} \sum_{j=1}^{N} (1 + \frac{j+1}{N})^2$

E) $R_N = \frac{1}{N} \sum_{j=0}^{N-1} (1 + (\frac{j+1}{N})^2)$

F) $R_N = \frac{2}{N} \sum_{j=0}^{N-1} (1 + \frac{j}{N})^2$

G) $R_N = \frac{2}{N} \sum_{j=0}^{N-1} (1 + \frac{j+1}{N})^2$

H) $R_N = \frac{1}{N} \sum_{j=0}^{N-1} (\frac{j+1}{N})^2$

I) $R_N = \frac{1}{N} \sum_{j=0}^{N-1} (1 + \frac{j}{N})^2$

→**J)** $R_N = \frac{1}{N} \sum_{j=0}^{N-1} (1 + \frac{j+1}{N})^2$

$$\Delta x = \frac{1}{N}, \quad a_j = 1 + \frac{j}{N}$$
$$R_N = \Delta x \sum_{j=0}^{N-1} f(a_{j+1}) = \frac{1}{N} \sum_{j=0}^{N-1} \left(1 + \frac{j+1}{N}\right)^2.$$

3. The limit

$$\lim_{N\to\infty} \frac{3}{N}\sum_{j=1}^{N}\left(2+\frac{3j}{N}\right)^4$$

represents a definite integral. What is this integral?

A) $\int_0^3 x^4 dx$

→ **B**) $\int_2^5 x^4 dx$

C) $\int_0^3 (2+3x)^4 dx$

D) $\int_0^3 (2+3x/N)^4 dx$

E) $\int_2^5 (x/N)^4 dx$

F) $\int_2^5 (2+x)^4 dx$

G) $\int_2^5 (2+3x)^4 dx$

H) $\int_1^N (2+3x/N)^4 dx$

I) $\int_0^1 (2+x)^4 dx$

J) $\int_0^2 (2+x)^4 dx$

The Riemann sum can be written in the form

$$\Delta x \sum_{j=1}^{N} f(a_j)$$

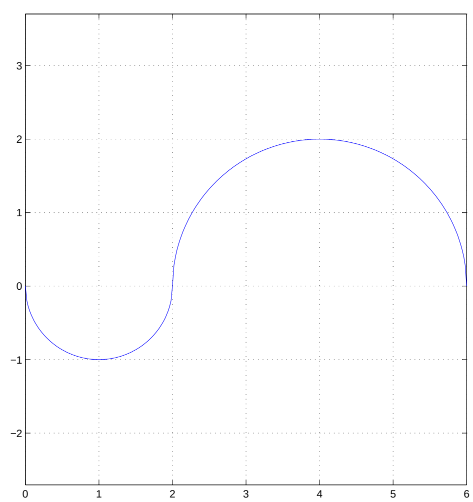
where $f(x) = x^4$ and the set of points $a_j = 2 + 3j/N,\ j = 1, \ldots, 30$, approximates the interval $[2, 5]$. Notice that $\Delta x = (5-2)/N = 3/N$. This corresponds to the integral

$$\int_2^5 x^4 dx.$$

4. Evaluate the sum of integrals

$$\int_0^2 f(x)dx + \int_0^6 |f(x)|dx$$

where $f(x)$ is the function shown in the graph.



A) $-\pi$

B) $-\pi/4$

C) $\pi/4$

D) $-\pi/2$

E) $\pi/2$

F) π

→**G**) 2π

H) 3π

I) $3\pi/2$

J) 4π

Notice that

$$\int_0^2 f(x)dx + \int_0^6 |f(x)|dx = \int_2^6 f(x)dx$$

is the area of the larger half-disc (of radius 2) which is 2π.

5. Calculate the integral

$$\int_0^5 (3f(x) - 5g(x))dx$$

assuming that $\int_0^5 f(x)dx = 7$ and $\int_0^5 g(x)dx = 5$.

A) -3

B) 3

→**C**) -4

D) 4

E) -7

F) 7

G) -2

H) 2

I) 5

J) -5

$$\begin{aligned}\int_0^5 (3f(x) - 5g(x))dx &= 3\int_0^5 f(x)dx - 5\int_0^5 g(x)dx \\ &= 3 \times 7 - 5 \times 5 \\ &= -4.\end{aligned}$$

6. Calculate the integral

$$\int_0^2 |x^2 - 1| dx.$$

A) -4

B) -3

C) -2

D) -1

E) 0

F) 1

→**G**) 2

H) 3

I) 4

J) 5

We have $x^2 - 1 < 0$ over the interval $(-1, 1)$, and positive for $x > 1$ and $x < -1$. Thus

$$\begin{aligned}
\int_0^2 |x^2 - 1| dx &= \int_0^1 (1 - x^2) dx + \int_1^2 (x^2 - 1) dx \\
&= (x - x^3/3)|_0^1 + (x^3/3 - x)|_1^2 \\
&= 1 - 1/3 + 8/3 - 2 - 1/3 + 1 \\
&= 2.
\end{aligned}$$

7. Calculate the integral

$$\int_{-2}^{0} (3x - 2e^x)\, dx.$$

A) $4e^2 - 2$

B) $2e^{-2} + 8$

C) $e^{-2} - 8$

D) $-e^{-2} + 8$

→**E**) $2e^{-2} - 8$

F) $2e^{-2} - 4$

G) $2e^2 - 8$

H) $2e^2 - 4$

I) $2e^2 + 8$

J) $4e^{-2} - 4$

$$\int_{-2}^{0} (3x - 2e^x)\, dx = (3x^2/2 - 2e^x)|_{-2}^{0} = -2 - (6 - 2e^{-2}) = 2e^{-2} - 8.$$

8. Calculate the integral

$$\int_a^{3a} \frac{dt}{t}.$$

A) $(\ln 2)/a$

B) $(\ln 3)/a$

C) $a \ln 3$

D) $a \ln 2$

E) $\ln 3/\ln a$

F) $\ln(3a)/\ln(a)$

G) $\ln(2a)$

H) $\ln(3/a)$

I) $\ln(3a)$

→**J**) $\ln 3$

$$\int_a^{3a} \frac{dt}{t} = \ln t|_a^{3a} = \ln(3a) - \ln a = \ln \frac{3a}{a} = \ln 3.$$

9. A function $G(s)$ is defined by the integral

$$G(s) = \int_{-6}^{\cos(s)} \left(u^4 - 3u\right) du.$$

Find $G'(s)$.

A) $G'(s) = \cos(s)(\cos^4(s) - 3\cos(s))$

B) $G'(s) = \cos(s)(\cos^4(s) - 3\cos(s)) + 6$

→**C**) $G'(s) = -\sin(s)(\cos^4(s) - 3\cos(s))$

D) $G'(s) = -\sin(s)(\cos^4(s) - 3\cos(s)) + 6$

E) $G'(s) = \sin(s)(\cos^4(s) - 3\cos(s))(4\cos^3(s)\sin(s) - 3\sin(s))$

F) $G'(s) = -\sin(s)(\cos^4(s) - 3\cos(s) + 6^4 - 18)$

G) $G'(s) = \cos(s)(\cos^4(s) - 3\cos(s) - 6^4 + 18)$

H) $G'(s) = -\sin(s)(\cos^4(s) - 3\cos(s) - 6^4 + 18)$

I) $G'(s) = \cos(s)(\sin^4(s) - 3\sin(s))$

J) $G'(s) = -\sin(s)(\cos^4(s) - 3\cos(s)) - 6$

Writing $F(x) = \int_{-6}^{x}(u^4 - 3u)du$ and $g(s) = \cos s$, we have $G(s) = F(g(s))$. The chain rule gives $G'(s) = F'(g(s))g'(s)$. Therefore,

$$G'(s) = (\cos^4 s - 3\cos s)(-\sin s).$$

10. Calculate the derivative

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2} \tan t dt.$$

A) $2x\sec^2(x^2) - \sec^2(\sqrt{x})/(2\sqrt{x})$
B) $\sec^2(x^2)/x^2 + \sec^2(\sqrt{x})\sqrt{x}$
C) $\tan(x^2)/2x - \tan(\sqrt{x})/\sqrt{x}$
D) $\sec^2(x^2)/x^2 + \tan(\sqrt{x})\sqrt{x}$
E) $\tan(x^2)/x^2 - \tan(\sqrt{x})\sqrt{x}$
F) $x^2\tan(x^2) - \tan(\sqrt{x})\sqrt{x}$
→**G**) $2x\tan(x^2) - \tan(\sqrt{x})/(2\sqrt{x})$
H) $2x\tan(x^2) - \tan(\sqrt{x})/(\sqrt{x})$
I) $x\tan(x^2) - \tan(\sqrt{x})/(2\sqrt{x})$
J) $-x\tan(x^2) + \tan(\sqrt{x})/(2\sqrt{x})$

$$\begin{aligned}\frac{d}{dx}\int_{\sqrt{x}}^{x^2} \tan t dt &= \frac{d}{dx}\int_0^{x^2} \tan t dt - \frac{d}{dx}\int_0^{\sqrt{x}} \tan t dt \\ &= 2x\tan(x^2) - \tan(\sqrt{x})/2\sqrt{x}.\end{aligned}$$

11. Water flows into an empty reservoir at a rate of $600 + 5t$ gallons per hour. What is the quantity of water in the reservoir after 2 hours?

A) 1207

B) 1208

C) 1209

→**D**) 1210

E) 1211

F) 1212

G) 1213

H) 1214

I) 1215

J) 1216

This quantity is given by the integral

$$\int_0^2 (600 + 5t)dt = 2 \times 600 + 5 \times 2^2/2 = 1200 + 10 = 1210.$$

12. The traffic flow rate past a certain point on a highway is

$$q(t) = 3000 + 2000t - 300t^2,$$

where t is in hours and $t = 0$ is 8 AM. How many cars pass by during the time interval from 8 to 10 AM?

A) 8400
B) 8600
C) 8800
D) 9000
→**E)** 9200
F) 9400
G) 9600
H) 9800
I) 10000
J) 10200

The number of cars is

$$\int_0^2 (3000 + 2000t - 300t^2)dt = 6000 + 4000 - 300 \times 8/3 = 9200.$$

13. Express the integral

$$\int_0^1 \frac{2x^2+x}{(4x^3+3x^2)^2}dx$$

as an integral in the variable u, using the substitution $u = 4x^3 + 3x^2$.

A) $\int_0^1 \frac{u}{u^2} du$

B) $\int_0^1 \frac{du}{u^2}$

C) $6\int_0^1 \frac{du}{u^2}$

D) $\frac{1}{6}\int_0^1 \frac{du}{u^2}$

E) $\int_0^7 \frac{du}{u^2}$

F) $6\int_0^7 \frac{du}{u^2}$

G) $\frac{1}{6}\int_0^3 \frac{du}{u^2}$

→**H)** $\frac{1}{6}\int_0^7 \frac{du}{u^2}$

I) $\int_0^3 \frac{du}{u^2}$

J) $\int_1^3 \frac{du}{u^2}$

If $u = 4x^3 + 3x^2$, then $du = 6(2x^2 + x)dx$, $u(0) = 0$ and $u(1) = 4 + 3 = 7$. Therefore,

$$\int_0^1 \frac{2x^2+x}{(4x^3+3x^2)^2}dx = \frac{1}{6}\int_0^7 \frac{du}{u}.$$

14. Find the definite integral

$$\int_0^{\ln 2} \frac{e^t}{e^{2t} + 2e^t + 1} dt.$$

A) 0

B) 6

→**C**) 1/6

D) $\ln 2$

E) $-1/\ln 2$

F) -6

G) $e^2 - 1$

H) $1 - e^2$

I) $6 - e^{-2}$

J) $e^2 - e^{-2}$

We use the substitution $u = e^t$. Then $du = e^t dt$, $u(0) = 1$, $u(\ln 2) = 2$, and

$$\begin{aligned}\int_0^{\ln 2} \frac{e^t}{e^{2t} + 2e^t + 1} dt &= \int_1^2 \frac{du}{u^2 + 2u + 1} \\ &= \int_1^2 \frac{du}{(1+u)^2} \\ &= \int_2^3 \frac{dv}{v^2} \\ &= (-v^{-1})|_2^3 \\ &= -1/3 + 1/2 \\ &= 1/6.\end{aligned}$$

15. Find the definite integral

$$\int_0^1 \theta \tan(\theta^2) d\theta.$$

A) $2\ln(\cos 1) - 1$

B) $-\frac{1}{2}\ln(\cos 1) + 1/2$

C) $-\ln(\cos 1)$

D) $-\frac{1}{2}\ln(\sin 1)$

→**E**) $-\frac{1}{2}\ln(\cos 1)$

F) $2\ln(\cos 1)$

G) $\ln(\cos 1)$

H) $2\ln(\tan 1) - 1$

I) $-\frac{1}{2}\ln(\tan 1)$

J) $-2\ln(\tan 1) - \tan 1$

We use the substitution $u = \cos\theta^2$. Thus $du = -2\theta\sin(\theta^2)d\theta$ and

$$\int_0^1 \theta \tan(\theta^2) d\theta = -\frac{1}{2}\int_1^{\cos 1} \frac{du}{u} = -\frac{1}{2}\ln(\cos 1).$$

16. (16 points)Find the following two indefinite integrals. In each case, clearly indicate the substitutions used.

(a)

$$\int xe^{-4x^2}\,dx$$

(b)

$$\int (\cos x)3^{\sin x}\,dx$$

For integral (a) we use the substitution $u = -4x^2$. So $du = -8xdx$ and

$$\int xe^{-4x^2}\,dx = -\frac{1}{8}\int e^u du = -\frac{1}{8}e^{-4x^2} + C.$$

For integral (b) we use the substitution $u = 3^{\sin x}$. So $du = \ln 3(\cos x)3^{\sin x}dx$ and

$$\int (\cos x)3^{\sin x}dx = \int \frac{du}{\ln 3} = u/\ln 3 + C = \frac{3^{\sin x}}{\ln 3} + C.$$

17. (9 points) Find all solutions to the differential equation

$$y' = -5y.$$

Which solution satisfies the initial condition $y(0) = 3.4$?

We know that the general solution of this equation is

$$y(t) = y_0 e^{-5t}.$$

The initial condition gives $y_0 = 3.4$. Under this condition the solution reduces to

$$y(t) = 3.4e^{-5t}.$$