This exam contains 15 multiple choice questions and 2 hand graded questions. The multiple choice questions are worth 5 points each and the hand graded questions are worth a total of 25 points. The latter questions will be evaluated not only for having the correct solutions but also for clarity. Points may be taken for confusing and disorganized writing, even when the answer is correct.

1. Find the area enclosed by the graphs of $f(x) = x$ and $g(x) = 2 - x^2$.

   A) 5/2
   B) 3
   C) 7/2
   D) 4
   →E) 9/2
   F) 5
   G) 11/2
   H) 6
   I) 13/2
   J) 7

   The points of intersection of the two graphs are obtained by setting $f(x) = g(x)$, or $x = 2 - x^2$. Equivalently, $x^2 + x - 2 = 0$. The roots are $x = 1$ and $x = -2$. The area is

$$\begin{aligned}\int_{-2}^{1} [g(x) - f(x)]dx &= \int_{-2}^{1} [2 - x^2 - x]dx \\ &= \left[2x - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-2}^{1} \\ &= 2 - \frac{1}{3} - \frac{1}{2} - \left(-4 + \frac{8}{3} - 2\right) \\ &= 8 - 3 - \frac{1}{2} \\ &= \frac{9}{2}.\end{aligned}$$

2. Calculate the volume of a cylinder inclined at an angle $\pi/6$ whose height is 10 and whose base is a circle of radius 4.

   A) $320\pi\pi$

   →B) 160π

   C) 120π

   D) 80π

   E) $80\sqrt{3}\pi$

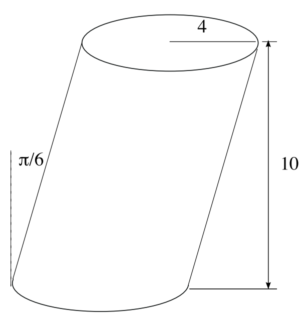
   F) $160\sqrt{3}\pi$

   G) $4\pi/3$

   H) 96π

   I) $96\sqrt{3}\pi$

   J) $320\sqrt{3}\pi$



The cross section of the cylinder at height x, for $0 \leq x \leq 10$, is a disc of radius 4, hence the cross-sectional area is constant, equal to $A(x) = \pi 4^2 = 16\pi$. The volume is given by the integral

$$\int_0^{10} A(x)dx = 16\pi \int_0^{10} dx = 160\pi.$$

3. Find the volume of the solid obtained by rotating the region enclosed by the graphs of $x = \sqrt{y}$ and $x = 0$ about the y-axis over the interval $1 \leq y \leq 3$.

   A) $\pi/4$
   B) $\pi/3$
   C) $\pi/2$
   D) π
   E) 2π
   F) 3π
   →G) 4π
   H) 8π
   I) 10π
   J) 12π

   By the disc method, the volume is

$$\begin{aligned}\int_1^3 \pi(\sqrt{y})^2\,dy &= \left[\frac{\pi y^2}{2}\right]_1^3 \\ &= \pi\frac{9-1}{2} \\ &= 4\pi.\end{aligned}$$

4. Compute the volume of the solid obtained by rotating the region enclosed by the graphs of $y = x^2$, $y = 2 - x^2$, and $x = 0$ about the y-axis.

   A) 8π
   B) 7π
   C) 6π
   D) 5π
   E) 4π
   F) 3π
   G) 2π
   →H) π
   I) $\pi/2$
   J) $\pi/4$

The intersection of the graphs of $y = x^2$ and $y = 2 - x^2$ happens for x such that $x^2 = 2 - x^2$. This equation has solutions $x = \pm 1$. The region bounded by these two graphs and $x = 0$ lies over the interval $0 \le x \le 1$. The volume, obtained by the shell method, is then

$$\begin{aligned}\int_0^1 2\pi x[2 - x^2 - x^2]dx &= 2\pi \int_0^1 x(2 - 2x^2)dx \\ &= 4\pi \int_0^2 (x - x^3)dx \\ &= 4\pi \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 \\ &= 4\pi \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= \pi.\end{aligned}$$

5. Calculate the volume of the solid obtained by rotating the region under the graph of $f(x) = x^{-2}$ over the interval $[-2, -1]$, about the axis of rotation $x = 4$.

A) $\pi(2 + \ln 2)/2$

B) $\pi(1 + \ln 2)$

C) $2\pi(1 + \ln 2)$

→D) $2\pi(2 + \ln 2)$

E) $\pi(4 + \ln 2)/5$

F) $3\pi(1 + \ln 5)$

G) $2\pi(2 + 3\ln 2)$

H) $5(2 + 3\ln 2)$

I) $5\pi(1 + \ln 2)/3$

J) $7(2 + \ln 2)$

Using the shell method we get:

$$\begin{aligned}
\int_{-2}^{-1} 2\pi(4-x)x^{-2}dx &= 2\pi \int_{-2}^{-1} (4-x)x^{-2}dx \\
&= 2\pi \int_{1}^{2} (4+u)u^{-2}du \quad \text{(doing a substitution } u = -x\text{)} \\
&= 2\pi \int_{1}^{2} (4u^{-2} + u^{-1})du \\
&= 2\pi \left[-4u^{-1} + \log u\right]_1^2 \\
&= 2\pi \left[-2 + \ln 2 + 4 - 0\right] \\
&= 2\pi(2 + \ln 2).
\end{aligned}$$

6. How much work is done raising a 4 kg mass to a height of 1 m above ground? (Recall: the acceleration of gravity is 9.8 meters per second squared.)

→A) 39.2 J
B) 38.2 J
C) 37.2 J
D) 36.2 J
E) 35.2 J
F) 34.2 J
G) 33.2 J
H) 32.2 J
I) 31.2 J
J) 30.2 J

The work is given by the product of 4 kg by 9.8 m/s^2 (the weight) by 1 m. So $W = 4 \times 9.8 \times 1 = 39.2$ J.

7. Compute the work required to stretch a spring from 1 to 2 centimeters past equilibrium, assuming that the spring constant is 200 kg per second squared.

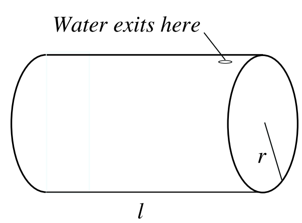
A) 0.36 J
B) 0.07 J
C) 0.15 J
D) 0.02 J
→E) 0.03 J
F) 0.43 J
G) 1.53 J
H) 2.03 J
I) 3.42 J
J) 4.50 J

The force function is $F(x) = 200x$, x is stretched from 0.01 m to 0.02 m. Thus the work is given by

$$\begin{aligned}\int_{0.01}^{0.02} 200x dx &= \left[100x^2\right]_{0.01}^{0.02} \\ &= 0.03 \text{ J.}\end{aligned}$$

8. Calculate the work in Joules required to pump all of the water out of the horizontal cylindrical tank shown in the figure. The cylinder has radius r and length l. The acceleration of gravity is $g = 9.8$ m per second squared and the density of water is $\rho = 1000$ kg per meter cubed. (Hint: You may need the integral $\int_{-r}^{r} \sqrt{r^2 - u^2} du = \pi r^2/2$.)

   A) $9.8\pi r^3 l$
   B) $98\pi r^3 l$
   C) $980 r^2 l^2$
   D) $9800 r^2 l^2$
   E) $9.8 r^2 l^2$
   F) $9800\pi r l^3$
   G) $9800\pi r^2 l^2$
   →H) $9800\pi r^3 l$
   I) $9800\pi r^3 l/3$
   J) $9800\pi r^3 l/6$



Let x be the height measured from the ground. Then the cross-section area is the area of a rectangle with sides l and $2\sqrt{r^2 - (r-x)^2}$, so

$$A(x) = 2l\sqrt{r^2 - (r-x)^2}.$$

The work to pump out the water (a distance $2r - x$ from level x) is then

$$\begin{aligned} g\rho \int_0^{2r} A(x)(2r-x)dx &= g\rho \int_0^{2r} 2l\sqrt{r^2-(r-x)^2}(2r-x)dx \\ &= 2lg\rho \int_{-r}^{r} (r+u)\sqrt{r^2-u^2}du \quad \text{using } u = r - x \\ &= 2lg\rho \left(r\int_{-r}^{r} \sqrt{r^2-u^2}du + \int_{-r}^{r} u\sqrt{r^2-u^2} \right) du \\ &= 2lg\rho \left(\frac{\pi r^3}{2} + 0 \right) \\ &= lg\rho\pi r^3. \end{aligned}$$

In the second to last line, the second integral is zero since the integrand is an odd function and the interval is symmetric about 0. The first integral is half the area of a disc of radius r. Thus the work is $W = 9800\pi r^3 l$.

9. Evaluate the integral $\int_0^\pi \sin(3x)\cos(4x)dx$.

   A) $3/2$
   B) $2\pi/3$
   C) $5/4$
   D) $3\pi/5$
   E) -3
   F) -2π
   G) $-5/7$
   H) $3/7$
   I) $\pi/5$
   →J) $-6/7$

We can use here the table of trigonometric integrals. It gives

$$\begin{aligned}\int_0^\pi \sin(3x)\cos(4x)dx &= \left[-\frac{\cos(3-4)x}{2(3-4)} - \frac{\cos(3+4)x}{2(3+4)}\right]_0^\pi \\ &= \left[\frac{\cos x}{2} - \frac{\cos(7x)}{14}\right]_0^\pi \\ &= -\frac{1}{2} + \frac{1}{14} - \left[\frac{1}{2} - \frac{1}{14}\right] \\ &= -\frac{6}{7}.\end{aligned}$$

10. Use the error bound $\text{Error}(S_N) \leq K_4(b-a)^5/(180N^4)$ to find a value of N such that the Simpson's rule approximation S_N for $\int_1^5 \ln x\, dx$ has an error of at most 10^{-6}. (Recall that in Simpson's rule N must be an even integer.)

    A) 605548
    B) 98
    C) 674
    D) 68
    E) 6
    F) 32
    G) 58
    →H) 78
    I) 74
    J) 78432

N must be an even integer such that

$$\frac{K_4(5-1)^5}{180N^4} \leq 10^{-6}.$$

To obtain K_4 we take the fourth derivative of $f(x) = \ln x$:

$$f'(x) = 1/x, \quad f''(x) = -1/x^2, \quad f^{(3)}(x) = 2/x^3, \quad f^{(4)}(x) = -6/x^4.$$

The maximum value of $|f^{(4)}(x)|$ over the interval $[1,5]$ is $K_4 = 6/1^4 = 6$. So we look for N such that

$$\frac{6(4^5)}{180N^4} \leq 10^{-6}.$$

Solving for N:

$$N \geq \left(\frac{6(4^5)10^6}{180}\right)^{1/4} = 76.4354$$

Rounding up to the next even integer gives

$$N = 78.$$

11. Evaluate the integral $\int_0^1 (2x+9)e^x dx$

 A) 8
 B) 4
 C) e^2
 D) $2e+4$
 E) $e+7$
 F) $3e-2$
 G) $2e^3+5$
 H) $5e^2-9$
 →I) $9e-7$
 J) $8e^3-5$

We can set $u(x) = 2x+9$ and $v'(x) = e^x$. Integrating by parts:

$$\begin{aligned}\int_0^1 (2x+9)e^x dx &= \left[(2x+9)e^x - 2\int e^x dx\right]_0^1 \\ &= [(2x+9)e^x - 2e^x]_0^1 \\ &= 11e - 9 - 2(e-1) \\ &= 9e - 7.\end{aligned}$$

12. Evaluate the integral $\int_0^{\pi/2} (x-1)\cos x \; dx$.

A) $\frac{\pi}{2} - 1$
B) $\pi - 2$
C) $2\pi - 1$
→D) $\frac{\pi}{2} - 2$
E) $\frac{\pi}{2} + 1$
F) 1
G) 2
H) 4
I) 0
J) $-\frac{\pi}{2}$

$$\begin{aligned}\int_0^{\pi/2} (x-1)\cos x dx &= \left[(x-1)\sin x - \int \sin x dx\right]_0^{\pi/2} \\ &= [(x-1)\sin x + \cos x]_0^{\pi/2} \\ &= \frac{\pi}{2} - 2.\end{aligned}$$

13. Evaluate the integral $\int_0^1 e^{\sqrt{x}} dx$

A) -1
B) 4
C) $2e^4 + 1$
D) $e + 1$
E) $2e + 4$
F) $3e + 2$
→G) 2
H) $3e$
I) $2e$
J) e

We use the substitution $u = \sqrt{x}$, or $x = u^2$. So $dx = 2udu$. Therefore,

$$\int_0^1 e^{\sqrt{x}} dx = \int_0^1 2ue^u du = 2\left[ue^u - e^u\right]_0^1 = 2.$$

14. Evaluate the integral $\int_0^{\pi/2} \sin^3\theta \cos^2\theta d\theta$

A) 2/21

B) 1/15

C) 2/9

D) 1/18

E) $2\pi/3$

F) 2π

G) $\pi/3$

→H) 2/15

I) $2\pi/15$

J) $3\pi/7$

$$\begin{aligned}
\int_0^{\pi/2} \sin^3\theta \cos\theta d\theta &= \int_0^{\pi/2} (1-\cos^2\theta)\cos^2\theta \sin\theta d\theta \\
&= \int_0^1 (1-u^2)u^2 du \\
&= \left[\frac{u^3}{3} - \frac{u^5}{5}\right]_0^1 \\
&= \frac{1}{3} - \frac{1}{5} \\
&= \frac{2}{15}.
\end{aligned}$$

15. Evaluate the integral $\int_0^2 \sqrt{4-x^2}dx$.

A) $\ln 2$
→B) π
C) 1
D) 2
E) 2π
F) 4π
G) $\sqrt{3}/2$
H) $\sin(4)$
I) $\sin(2)$
J) $2\pi/3$

We use the substitution $x = 2\sin\theta$, so $dx = 2\cos\theta d\theta$. Then

$$\begin{aligned}\int_0^2 \sqrt{4-x^2}dx &= 4\int_0^{\pi/2} \cos^2\theta d\theta \\ &= 2\int_0^{\pi/2} (1+\cos(2\theta))d\theta \\ &= 2\left[\theta + \frac{\sin(2\theta)}{2}\right]_0^{\pi/2} \\ &= 2\left(\frac{\pi}{2}+0\right) \\ &= \pi.\end{aligned}$$

16. (12 points) Evaluate the indefinite integral $\int x^3 \ln x \; dx$.

We use integration by parts, with $u'(x) = x^3$ and $v(x) = \ln x$. Then $u(x) = x^4/4$ and $v'(x) = 1/x$. Therefore,

$$\begin{aligned}\int x^3 \ln x dx &= \int u'(x)v(x)dx \\ &= u(x)v(x) - \int u(x)v'(x)dx \\ &= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}dx \\ &= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx \\ &= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C.\end{aligned}$$

17. (13 points) Find the approximations T_2, M_2 and S_4 for the integeral $\int_0^2 x^2 dx$.

For both T_2 and M_2, we have $\Delta x = 1$. Then the trapezoidal approximation is

$$\begin{aligned} T_2 &= \Delta x \left(\frac{f(0)+f(1)}{2} + \frac{f(1)+f(2)}{2} \right) \\ &= \frac{0+1}{2} + \frac{1+4}{2} \\ &= 3. \end{aligned}$$

The midpoint approximation is

$$\begin{aligned} M_2 &= \Delta x \left(f(1/2) + f(3/2) \right) \\ &= \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 \\ &= \frac{5}{2}. \end{aligned}$$

The Simpson S_4 approximation can now be obtained as

$$S_4 = \frac{1}{3}T_2 + \frac{2}{3}M_2 = \frac{1}{3}3 + \frac{2}{3}\frac{5}{2} = \frac{8}{3}.$$

Notice that $\frac{8}{3}$ is the exact value of the integral. This is to be expected since the Simpson rule is exact for polynomials up to order 3.