This exam contains 15 multiple choice questions and 2 hand graded questions. The multiple choice questions are worth 5 points each and the hand graded questions are worth a total of 25 points. The latter questions will be evaluated not only for having the correct solutions but also for clarity. Points may be taken for confusing and disorganized writing, even when the answer is correct.

1. Evaluate the integral

$$\int_4^5 \frac{2x-1}{x^2-5x+6}dx$$

A) $3\ln(3/2)$
B) $5\ln 2$
C) $\ln(2/3)+5\ln 3$
D) $3\ln(3)+4\ln 2$
E) $5\ln(2)+3\ln 5$
F) $\ln(2)+5$
*G) $3\ln(2/3)+5\ln 2$
H) $3+5\ln 2$
I) $3\ln(5)$
J) $4\ln(2/3)+7\ln 2$

First note that $x^2-5x+6=(x-2)(x-3)$. We can expand the integrand into partial fractions by solving for A and B:

$$\frac{2x-1}{x^2-5x+6}=\frac{A}{x-2}+\frac{B}{x-3}.$$

Writing both sides over a common denominator and equating the numerators gives:

$$2x-1=A(x-3)+B(x-2).$$

This is easily solved and gives: $A=-3$, $B=5$. Therefore,

$$\begin{aligned}\int_4^5 \frac{2x-1}{x^2-5x+6}dx &= \int_4^5 \left(\frac{-3}{x-2}+\frac{5}{x-3}\right)dx \\ &= \left[-3\ln|x-2|+5\ln|x-3|\right]_4^5 \\ &= 3\ln(2/3)+5\ln 2.\end{aligned}$$

2. Evaluate the integral

$$\int_1^2 \frac{dx}{x(x^2+1)}dx$$

A) $\ln\left(\frac{2}{\sqrt{5}}\right)$

B) $\ln\left(\frac{1}{\sqrt{2}}\right)$

C) $\ln\left(\frac{3}{\sqrt{2}}\right) - \ln\left(\frac{5}{\sqrt{2}}\right)$

D) $\ln\left(\frac{3}{\sqrt{2}}\right)$

E) $\ln\left(\frac{7}{\sqrt{3}}\right) - \ln\left(\frac{1}{\sqrt{5}}\right)$

*F) $\ln\left(\frac{2}{\sqrt{5}}\right) + \ln\left(\sqrt{2}\right)$

G) $\ln\left(\frac{5}{\sqrt{5}}\right)$

H) $\ln\left(\frac{7}{\sqrt{2}}\right)$

I) $\ln\left(\frac{2}{\sqrt{7}}\right) + \ln\left(\frac{3}{\sqrt{2}}\right)$

J) $\ln 3$

The partial fractions decomposition has the form

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}.$$

Equating the numerators after setting both sides over the same denominator:

$$A(x^2+1) + (Bx+C)x = 1.$$

Equivalently,

$$(A+B)x^2 + Cx + A = 1.$$

This gives $B = -A$, $C = 0$ and $A = 1$. Therefore,

$$\begin{aligned}\int_1^2 \frac{1}{x(x^2+1)}dx &= \int_1^2 \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx \\ &= \left[\ln|x| - \ln\sqrt{1+x^2}\right]_1^2 \\ &= \ln(2/\sqrt{5}) + \ln(\sqrt{2}).\end{aligned}$$

3. Determine whether the following improper integrals converge or diverge:

$$(a)\int_1^\infty \frac{dx}{x^{19/20}}, \quad (b)\int_0^5 \frac{dx}{x^{20/19}}, \quad (c)\int_{-3}^\infty \frac{dx}{(x+4)^{3/2}}, \quad (d)\int_0^1 \frac{dx}{x}.$$

(Below, c stands for 'converges' and d for 'diverges.')

A) c, c, c, c
B) d, c, d, d
*C) d, d, c, d
D) c, c, c, d
E) c, d, c, c
F) d, d, d, d
G) c, c, d, c
H) c, c, d, d
I) d, c, c, d
J) c, d, d, d

Recall that if a is a finite positive number, the integral $\int_0^a \frac{dx}{x^p}$ converges for $p <$ and diverges for $p \geq 1$. Therefore, both (b) and (d) diverge. The integral $\int_a^\infty \frac{dx}{x^p}$ converges for $p > 1$ and diverges for $p \leq 1$. Therefore, (a) diverges. The integral (c) is equal to $\int_1^\infty \frac{du}{u^{3/2}}$ (this is seen by doing a substitution $u = x + 4$), which converges.

4. Determine whether the integral

$$I = \int_0^1 x \ln x \, dx$$

is convergent or not. If it is convergent, evaluate it.

A) convergent, $I = 1/8$
B) convergent, $I = 1/2$
C) convergent, $I = 1$
D) convergent, $I = -1/2$
E) convergent, $I = -1$
F) convergent, $I = 2$
*G) convergent, $I = -1/4$
H) divergent
I) convergent, $I = 4$
J) convergent, $I = -4$

$I = \lim_{a \to 0^+} \int_a^1 x \ln x \, dx$. Integration by parts gives:

$$\begin{aligned} \int_a^1 x \ln x \, dx &= \left[\frac{x^2}{2} \ln x - \int \frac{x}{2} \right]_a^1 \\ &= \left[\frac{x^2}{2} \ln x - \frac{x^2}{4} \right]_a^1 \\ &= -\frac{1}{4} - \frac{a^2 \ln a}{2} + \frac{a^2}{4}. \end{aligned}$$

It can be shown using L'Hôspital's rule that $\lim_{a \to 0^+} a \ln a = 0$. From this limit it immediately follows that $\lim_{a \to 0^+} a^2 \ln a = 0$. Therefore, the limit as $a \to 0$ exists and is equal to $-1/4$.

5. Calculate the arc length of the graph of $y = x^{3/2}$ over the interval $[1, 2]$.

A) $2e + \sqrt{7}$

B) $\frac{7}{27}\left[\left(\frac{5}{2}\right)^{5/2} - \left(\frac{3}{4}\right)^{5/2}\right]$

*C) $\frac{8}{27}\left[\left(\frac{11}{2}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right]$

D) $\frac{8}{3}\left[\left(\frac{11}{2}\right)^{5/2} - \left(\frac{11}{4}\right)^{5/2}\right]$

E) $\frac{8}{3}\left[\left(\frac{9}{2}\right)^{1/2} - \left(\frac{7}{4}\right)^{1/2}\right]$

F) $\frac{1}{4}\left[\left(\frac{13}{2}\right)^{3/2} - \left(\frac{15}{4}\right)^{1/2}\right]$

G) $\frac{1}{3}\left[\left(\frac{7}{2}\right)^{3/2} - \left(\frac{17}{4}\right)^{1/2}\right]$

H) $\frac{3}{19}\left[\left(\frac{15}{2}\right)^{3/2} - \left(\frac{11}{4}\right)^{3/2}\right]$

I) $\frac{8}{9}\left(\frac{13}{4}\right)^{3/2}$

J) $\frac{8}{27}\left(\frac{11}{2}\right)^{3/2}$

The arclength I is given by

$$I = \int_1^2 \sqrt{1 + f'(x)^2}\, dx = \int_1^2 \sqrt{1 + \frac{9}{4}x}\, dx.$$

The change of variables $u = 1 + 9x/4$ gives

$$I = \frac{4}{9}\int_{1+\frac{9}{4}}^{1+\frac{9}{2}} u^{1/2} du = \frac{8}{27}\left[u^{3/2}\right]_{13/4}^{11/2} = \frac{8}{27}\left[(11/2)^{3/2} - (13/4)^{3/2}\right].$$

6. Compute the surface area of revolution defined by the function $y = x + 1$ over the interval $[0, 1]$.

   A) $2\sqrt{2}\pi$
   B) $5\sqrt{2}\pi$
   C) $7\sqrt{2}\pi$
   D) $3\sqrt{5}\pi$
   E) $7\sqrt{5}\pi$
   F) $2\sqrt{5}\pi$
   G) $8\sqrt{2}\pi$
   *H) $3\sqrt{2}\pi$
   I) $3\sqrt{7}\pi$
   J) $5\sqrt{3}\pi$

The surface area is given by

$$\begin{aligned}
2\pi \int_0^1 f(x)\sqrt{1 + f'(x)^2}\, dx &= 2\pi \int_0^1 (x+1)\sqrt{1+1^2}\, dx \\
&= 2\sqrt{2}\pi \int_0^1 (x+1) dx \\
&= 2\sqrt{2}\pi \left[\frac{x^2}{2} + x\right]_0^1 \\
&= 2\sqrt{2}\pi 3/2 \\
&= 3\sqrt{2}\pi.
\end{aligned}$$

7. Which of the following integrals correctly represents the surface area of revolution obtained by rotating the graph of $y = \sin x$ about the x-axis over the interval $[0, \pi]$?

A) $\int_0^\pi \sin x\sqrt{1+\cos^2 x}\ dx$

B) $\pi \int_0^\pi \sin x\sqrt{1+\cos^2 x}\ dx$

C) $2\pi \int_0^\pi \cos x\sqrt{1+\sin^2 x}\ dx$

*D) $2\pi \int_0^\pi \sin x\sqrt{1+\cos^2 x}\ dx$

E) $2\pi \int_0^\pi \sqrt{1+\cos^2 x}\ dx$

F) $\pi \int_0^\pi \sqrt{1+\cos^2 x}\ dx$

G) $\int_0^\pi \sqrt{1+\cos^2 x}\ dx$

H) $2\pi \int_0^\pi \cos x\sqrt{1+\cos^2 x}\ dx$

I) $2\pi \int_0^\pi \sin x\sqrt{1+\sin^2 x}\ dx$

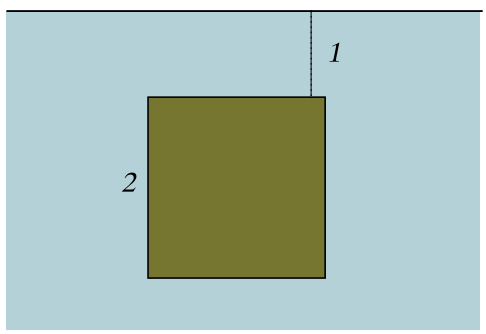
J) $4\pi \int_0^\pi \cos x\sqrt{2+\sin^2 x}\ dx$

The general integral expression for the area of a surface of revolution is $I = 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\ dx$. Therefore, if $f(x) = \sin x$, we have

$$I = 2\pi \int_0^\pi \sin x\sqrt{1+\cos^2 x}\ dx.$$

8. If w denotes the weight density of water, find the fluid force on a submerged vertical square plate of side 2 meters having its top side at a depth of 1 meter.

   A) w
   B) $2w$
   C) $4w$
   *D) $8w$
   E) $3w$
   F) $5w$
   G) $7w$
   H) $9w$
   I) $w/2$
   J) $w/4$



We fix the y-axis with origin at a depth of 1 meter pointing down. Therefore, the top side of the square is at $y = 0$ and the bottom side is at $y = 2$. The width at level y is constant, equal to $l(y) = 2$, the depth of a point associated to coordinate y is $1 + y$, and the pressure at level y is $p(y) = w(1 + y)$. So the force is obtained by

$$F = \int_0^2 p(y)l(y)\ dy = 2w \int_0^2 (1 + y)dy = 2w[2 + 2^2/2] = 8w.$$

9. Find the area and the x-coordinate of the centroid of the region lying between the graphs of $y = x^2$ and $y = \sqrt{x}$ over the interval $[0, 1]$.

A) $A = 1/6$, $x_{CM} = 5/9$
*B) $A = 1/3$, $x_{CM} = 9/20$
C) $A = 1/6$, $x_{CM} = 5/11$
D) $A = 1/3$, $x_{CM} = 5/16$
E) $A = 1/6$, $x_{CM} = 5/13$
F) $A = 1/6$, $x_{CM} = 3/5$
G) $A = 1/3$, $x_{CM} = 9/10$
H) $A = 1/6$, $x_{CM} = 3/7$
I) $A = 1/6$, $x_{CM} = 3/10$
J) $A = 1/3$, $x_{CM} = 3/10$

The area is

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \left[\frac{2x^{3/2}}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

The x-coordinate of the centroid is

$$x_{CM} = \frac{1}{A}\int_0^1 x\left(\sqrt{x} - x^2\right) dx = 3\left[\frac{2x^{5/2}}{5} - \frac{x^4}{4}\right]_0^1 = 3\left[\frac{2}{5} - \frac{1}{4}\right] = \frac{9}{20}.$$

10. Find the area and the x-coordinate of the centroid of the quarter of the unit disc centered at the origin $(0,0)$ and lying in the first quadrant.

A) $A = \pi/2$, $x_{CM} = \frac{4}{\pi}$
B) $A = \pi/3$, $x_{CM} = \frac{4}{\pi}$
C) $A = \pi/4$, $x_{CM} = \frac{4}{\pi}$
D) $A = \pi/4$, $x_{CM} = \frac{8}{3\pi}$
*E) $A = \pi/4$, $x_{CM} = \frac{4}{3\pi}$
F) $A = \pi/2$, $x_{CM} = \frac{2}{\pi}$
G) $A = \pi/2$, $x_{CM} = \frac{3}{\pi}$
H) $A = \pi/4$, $x_{CM} = \frac{8}{\pi}$
I) $A = \pi/4$, $x_{CM} = \frac{7}{3\pi}$
J) $A = \pi$, $x_{CM} = \frac{3}{\pi}$

The x-coordinate of the centroid equals, by symmetry, the y-coordinate, and the latter is given by: of the region between $y = 0$ and

$$x_{CM} = \frac{1}{2A}\int_0^1 f(y)^2 dy$$

where in this case $f(y) = \sqrt{1-y^2}$. The area of the quarter disc of radius 1 is $A = \pi r^2/4 = \pi/4$. Therefore,

$$x_{CM} = \frac{2}{\pi}\int_0^1 (1-y^2)\ dy = \frac{2}{\pi}\left[y - \frac{y^3}{3}\right]_0^1 = \frac{2}{\pi}\left[1 - \frac{1}{3}\right] = \frac{4}{3\pi}.$$

11. Calculate the Taylor polynomial $T_2(x)$ at $a = 0$ for the function

$$f(x) = \frac{1}{1+x}.$$

A) $T_2(x) = -x + 2x^2$
B) $T_2(x) = 1 + x^2$
C) $T_2(x) = 1 - x$
D) $T_2(x) = 2 - x + x^2$
E) $T_2(x) = 1 - 2x + 3x^2$
F) $T_2(x) = 3 - 2x + x^2$
G) $T_2(x) = 1 - x + 6x^2$
H) $T_2(x) = 6 - 3x + 2x^2$
*I) $T_2(x) = 1 - x + x^2$
J) $T_2(x) = 1 - 4x$

The first two derivatives of $f(x) = 1/(1+x)$ are $f'(x) = -1/(1+x)^2$ and $f''(x) = 2/(1+x)^3$. At $x = 0$, $f(0) = 1$, $f'(0) = -1$ and $f''(0) = 2$. So the coefficients of the Taylor polynomial $T_2(x)$ are 1, -1, and $2/2! = 1$. Therefore,

$$T_2(x) = 1 - x + x^2.$$

12. Find an error bound E for approximating $\sin x$ by the Maclaurin polynomial $T_4(x) = x - x^3/6$ over the interval $[-\pi, \pi]$. I.e., find E so that

$$|\sin x - T_4(x)| \leq E$$

over the interval. (An error bound that is much bigger than the optimal one will be considered wrong.)

*A) $\pi^5/120$
B) $\pi^4/24$
C) $\pi^5/24$
D) $1/120$
E) 0
F) $2\pi^5/15$
G) $4\pi^5/15$
H) $8\pi^5/15$
I) $1/24$
J) $5\pi/120$

The general expression for the error bound for $R_4(x)$ is

$$|R_4(x)| \leq \frac{K|x|^5}{5!}$$

where K is an upper bound for the derivative of order 5 over the given interval. The absolute value of the fifth derivative of $\sin x$ is less than or equal to 1 for all x. So we can take $K = 1$. So $|R_4(x)|$ is at most $|x|^5/5! = |x|^5/120$. Over the interval $[-\pi, \pi]$ the quantity $|x|$ is at most π. Therefore, the error bound we are looking for is $\pi^5/120$.

13. Solve the initial value problem

$$y' = xy^2, \quad y(0) = -1.$$

A) $y(x) = 1/(x^3 - 2)$
B) $y(x) = -5/(x^2 - 5)$
C) $y(x) = 2/(x - 2)$
D) $y(x) = -1/(x^4 - 2)$
E) $y(x) = 2/(x^4 - 4)$
F) $y(x) = 1/(x - 4)$
G) $y(x) = -6/(x^3 - 3)$
H) $y(x) = 1/(x^4 - 1)$
I) $y(x) = 1/(x - 5)$
*J) $y(x) = -2/(x^2 + 2)$

This is a separable equation, which we can solve by writing

$$\int \frac{dy}{y^2} = \int x \, dx.$$

Therefore,

$$-y^{-1} = \frac{x^2}{2} + C.$$

The initial condition $y(0) = -1$ gives $1 = C$, so $-y^{-1} = 1 + x^2/2$. We can now solve for y in terms of x:

$$y(x) = -\frac{1}{1 + \frac{x^2}{2}} = 1\frac{2}{x^2 + 2}.$$

14. Find all values of a such that $y = e^{ax}$ is a solution of

$$y'' + 2y' - 8y = 0.$$

A) $1, 2$
B) $2, -1$
C) $1, 4$
D) $2, 4$
E) $1, 3$
F) $2, 3$
*G) $2, -4$
H) $2, -3$
I) $3, -3$
J) $1, 0$

Substituting e^{ax} for y into the differential equation gives:

$$e^{ax}(a^2 + 2a - 8) = 0.$$

Therefore a must be a root of the algebraic equation $x^2 + 2x - 8$. The two roots are 2 and -4.

15. The velocity v of a skydiver can be determined using the differential equation

$$v' = -\frac{10}{m}\left(v + \frac{gm}{10}\right)$$

where m is the diver's mass, $g = 9.8\ \mathrm{m/s^2}$ is the acceleration due to gravity. If a 60-kg skydiver jumps out of an airplane, what is her terminal velocity in meters per second?

*A) -58.8 m/s
B) -49.9 m/s
C) -52.3 m/s
D) -38.8 m/s
E) -76.8 m/s
F) -34.9 m/s
G) -82.5 m/s
H) -35.8 m/s
I) -25.8 m/s
J) -39.1 m/s

The terminal velocity is obtained from the condition $v' = 0$, so $v = -gm/10$. Therefore,

$$v = -\frac{9.8 \times 60}{10} = -58.8$$

16. (12 points) The following two question refer to the improper integral

$$I = \int_1^\infty \frac{1}{\sqrt{x^5+2}}dx.$$

(a) Does the integral converge or diverge?

(b) Explain how the Comparison Test for improper integrals is used to answer the first question.

(a) The integral converges.

(b) We can use the comparison test as follows. First note that

$$0 \leq \frac{1}{\sqrt{x^5+2}} \leq \frac{1}{x^{5/2}}.$$

But as $p = 5/2 > 1$, the integral

$$\int_1^\infty \frac{1}{x^{5/2}}dx$$

converges. Therefore, the integral I must also converge.

17. (13 points) (Rogawski 8.4 # 37) Let $T_n(x)$ denote the Taylor polynomial of the function $f(x) = \ln x$ at $a = 1$.

    (a) Find $T_3(x)$.

    (b) Using the error bound for $R_3(x)$, show that

$$|\ln(1.3) - T_3(1.3)| \leq 2 \times 10^{-3}.$$

(a) The derivatives of $f(x) = \ln x$ are

$$f^{(1)}(x) = x^{-1},\ f^{(2)}(x) = -x^{-2},\ f^{(3)}(x) = 2x^{-3}.$$

Evaluated at $a = 1$, gives

$$f(1) = 0,\ f^{(1)}(1) = 1,\ f^{(2)}(1) = -1,\ f^{(3)}(1) = 2.$$

Therefore,

$$T_3(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3}.$$

(b) The fourth derivative of $f(x)$ is $f^{(4)}(x) = -6x^{-4}$. The constant $K = 6$ is an upper-bound for $|f^{(4)}(x)|$ for all x in the interval $[1, 1.3]$. Therefore,

$$|R_4(1.3)| \leq \frac{K \times 0.3^5}{4!} = 6 \times 0.0081/24 = 0.002$$