An Introduction to Cocycle Super-Rigidity

Renato Feres

Washington University, St. Louis MO 63130, USA

Abstract. The cocycle super-rigidity theorem is a central result in the study of dynamics of semisimple Lie groups and lattices. We give an overview of the main ideas centered on this theorem and some of its most immediate applications. The emphasis will be on the topological and differentiable (as opposed to measurable) aspects of the theory.

1 Introduction

The dynamical study of actions of semisimple Lie groups is a subject of present research whose sources and motivations come from a wide range of topics such as the geometry and topology of spaces of non-positive curvature, linear representation of lattice groups, the theory of random walks on groups, to cite a few. Of central importance to this field are the celebrated super-rigidity and arithmeticity theorems of Margulis. These theorems are fundamental for describing the structure and linear representations of lattice subgroups of semisimple Lie groups (see [8], as well as [9]). The super-rigidity theorem was extended into the nonlinear setting of G-spaces by Zimmer, whose cocycle super-rigidity theorem is now an essential tool for the development of the ergodic theory of actions of semisimple groups and their lattice subgroups (cf. [11]).

The purpose of these notes is to provide a brief overview of some of the main ideas centered on the cocycle super-rigidity theorem and some of the connections this theorem has with differential geometry and dynamical systems. Differently from [11], which is concerned mainly with measurable ergodic theory, we will emphasize the topological and differentiable aspects of the theory, along the lines of [5].

Sections 1, 2 and 3 give some motivation for the main theorem by showing how the problem of looking for invariant "geometric structures" for a given group action is related to 1-cocycles over the action. Section 4 contains an overview of basic results about semisimple Lie groups that will be needed later while section 5 reviews the necessary points of dynamics and ergodic theory. The main result – the topological super-rigidity theorem – is stated in section 7 after some preparatory material on algebraic hulls given in section 6. Section 8 contains a detailed sketch of the proof (of a simplified but representative statement). Finally in section 9 we discuss a few of the more immediate applications and briefly indicate other related results and applications.

In the theory of dynamical systems it is generally the case that the degree of regularity (of invariant structures, conjugacies, etc.) is a central concern.

action of a group
random action

For the main result of these notes, however, regularity does not play an important role in the sense that the conclusion will be valid in the C^s class (where s can be understood as 'measurable', 'Hölder continuous', 'smooth', 'real analytic', etc.) as long as the hypothesis are valid in the same class. In this way the topological super-rigidity theorem may be viewed as a 1-parameter theorem. The reader should keep this in mind whenever coming across an expression of the sort: "let φ be, say, a continuous map..."

2 Cocycles Over Group Actions

We describe here some dynamical problems and situations where the idea of cocycles over group actions arise naturally.

2.1 Random Actions

The idea of a random action will be used here to motivate the definition of a (measurable) cocycle over a group action, although this particular interpretation is not important for the sequel. (The interested reader will find much more about this idea in [1].)

An action of a group G on a set V is by definition a map $\Phi : G \times V \to V$ that satisfies the properties:

1. Each $\Phi_g := \Phi(g, \cdot)$ is an invertible self-map of V and $\Phi_{g^{-1}} = \Phi_g^{-1}$
2. $\Phi_{g_1 g_2} = \Phi_{g_1} \circ \Phi_{g_2}$ for all $g_1, g_2 \in G$.

(Later on, we will often write simply gv instead of $\Phi_g(v)$.)

An invertible transformation $f : V \to V$ of a (measurable, topological, differentiable, etc.) space V gives rise, by iteration, to a $\mathbb{Z}$-action on V:

$$\Phi(n, u) = \begin{cases} \underbrace{f \circ \cdots \circ f}_{n}(u) & \text{if } n > 0 \\ \underbrace{f^{-1} \circ \cdots \circ f^{-1}}_{n}(u) & \text{if } n < 0 \\ u & \text{if } n = 0\,. \end{cases}$$

For $\mathbb{Z}$ (or for $\mathbb{R}$) the group parameter is naturally thought to represent time, so that $n \mapsto \Phi_n(u)$ describes the *orbit* of a point $u \in V$ under the time evolution defined by Φ. If $G = \mathbb{R}$ and Φ is smooth (on a smooth manifold V), then the action arises by integrating a vector field on V and Φ_t is the flow of that vector field. For bigger groups it is no longer natural to think of G as the "time" of a dynamical system, although a fair amount of the concepts and results from the theory of one-parameter dynamics (and ergodic theory) carries on to this more general situation.

An interesting generalization of the concept of group action is that of a *random action*. Before giving a definition, we consider the following elementary example. Let $f_0, f_1 : V \to V$ be invertible self-maps of V. The map f_i,

$i = 0, 1$, is given a probability q_i and the iteration proceeds as follows: flip a (possibly biased) coin with probabilities $P(\text{Head}) = q_0, P(\text{Tail}) = q_1$ an infinite ($\mathbb{Z}$) number of times and register the outcome as a sequence of 0s and 1s:

shift map
Bernoulli measure
cocycle identity
cocycle over a group action

$$x = (\cdots x_{-1}\, x_0\ x_1 \cdots)\,.$$

The orbit of a point $u \in V$ is now given by:

$$\Phi(n,x)(u) = \begin{cases} f_{x_{n-1}} \circ \cdots \circ f_{x_0}(u) & \text{if } n > 0 \\ f^{-1}_{x_{-n}} \circ \cdots \circ f^{-1}_{x_{-1}}(u) & \text{if } n < 0 \\ u & \text{if } n = 0\,. \end{cases}$$

To see what replaces the homomorphism property $\Phi_{n_1+n_2} = \Phi_{n_1} \circ \Phi_{n_2}$ in this random case, we will need the *shift map*, $\sigma : M \to M$, on the space of sequences

$$M := \{0,1\}^{\mathbb{Z}} = \{x = (\cdots x_{-1}\, x_0\ x_1 \cdots) : x_i \in \{0,1\}\}\,.$$

The shift map is defined by

$$\sigma(x) = x',\ \text{where } x'_i = x_{i+1} \text{ for each } i.$$

If M is given the product probability measure (the *Bernoulli measure*) that assigns probability q_k to the event $\{x \in M : x_i = k\}$, $i \in \mathbb{Z}$ and $k \in \{0,1\}$, then σ becomes a measure preserving invertible transformation of M and M itself is now endowed with a $\mathbb{Z}$-action generated by σ.

We can now write, for all $n, m \in \mathbb{Z}$ and $x \in M$:

$$\Phi(n+m, x) = \Phi(n, \sigma^m(x)) \circ \Phi(m, x)\,. \tag{1}$$

General Groups; Cocycles. The definition of a random $\mathbb{Z}$-action readily generalizes to other groups. Let G be a group and M a set (say, a probability space) on which G acts. We denote the action by $\sigma : G \times M \to M$. Let V be, say, a manifold and consider a group H of homeomorphisms of V (possibly preserving further "structure" on V). A random action of G on V is now a map $c : G \times M \to H$ that satisfies 1, which we write in this case as

$$c(g_1 g_2, x) = c(g_1, \sigma(g_2, x))c(g_2, x) \tag{2}$$

for all $g_1, g_2 \in G$ and all $x \in M$. Property 2 characterizes a *cocycle over the G-action* σ, taking values in H (regardless of whether H is viewed as a group of transformations of another space V or whether a G-invariant structure of some sort is imposed on M).

An action of G on V is clearly a cocycle over the trivial action of G on a one-point set. A more general, but still essentially "non-random," situation is given as follows. Let $\varphi : M \to H$ be simply a function and $\rho : G \to H$ a group

cohomologous cocycles
rho-simple cocycle

homomorphism into a group H (for example, of invertible transformations of V, so that ρ defines a G-action on V). It is an easy exercise to verify that if $\sigma : G \times M \to M$ is a G-action then

$$c(g,x) = \varphi(\sigma(g,x))^{-1}\rho(g)\varphi(x) \tag{3}$$

is a cocycle over σ. (This may be interpreted as changing the action on V given by ρ by a "random coordinate change" on V specified by φ.) We say in this case that c is *cohomologous* to the cocycle defined by $(g,x) \mapsto \rho(g)$. This latter cocycle will be said to be ρ-*simple*.

2.2 What Is It Good For?

Many problems in dynamics can be formulated in terms of proving that a cocycle over a certain group action is cohomologous to a ρ-simple cocycle. We give next a few examples.

Invariant Measures. If an action σ of a group G on M preserves a measure class, represented by a not necessarily invariant probability measure, μ, then we obtain a cocycle into the multiplicative group of positive real numbers given by the Radon–Nikodym derivative (we use here the simpler notation $\sigma_g(x) = gx$)

$$J(g,x) := \frac{d(g_*^{-1}\mu)}{d\mu}(x)\,,$$

where $g_*\mu$ denotes the natural action of G on measures, defined by

$$(g_*\mu)(A) := \mu(g^{-1}A) \tag{4}$$

for any measurable set $A \subset M$. The cocycle property can be checked as follows. Let f be any measurable bounded function on M. Then

$$\begin{aligned}
\int f(x)J(g_1,g_2x)J(g_2,x)\;d\mu(x) &= \int f(x)J(g_1,g_2x)\;d((g_2^{-1})_*\mu)(x)\\
&= \int f(g_2^{-1}x)J(g_1,x)\;d\mu(x)\\
&= \int f(g_2^{-1}x)\;d({g_1^{-1}}_*\mu)(x)\\
&= \int f(g_2^{-1}(g_1^{-1}))\;d\mu(x)\\
&= \int f((g_1g_2)^{-1}(x))\;d\mu(x)\\
&= \int f(x)J(g_1g_2,x)\;d\mu(x)\,.
\end{aligned}$$

Radon–Nikodym cocycle

nonstationary linearizations

Since f is arbitrary we get (a.e.)

$$J(g_1 g_2, x) = J(g_1, g_2 x) J(g_2, x) \,.$$

The conditions for the measure class defined by μ to contain a G-invariant measure is that the Radon–Nikodym cocycle be cohomologous to the trivial cocycle. Note that if $J(g,x) = \varphi(\sigma_g(x))^{-1}\varphi(x)$ for some measurable function φ on M, then the measure ν such that

$$\frac{d\nu}{d\mu}(x) = \varphi(x)$$

is invariant, as a calculation similar to the previous one shows. The converse is also immediate to prove.

Nonstationary Linearizations. As another example of a situation that naturally leads to the study of cocycles over a group action we mention the problem of finding *nonstationary linearizations*. We first introduce a few definitions. Let $\mathcal{J}$ denote the set of all local parametrizations of a smooth manifold M of dimension n. By definition, $\varphi \in \mathcal{J}$ is a diffeomorphism from a neighborhood of the origin, $\mathcal{D}_\varphi$, to an open set in M. (For simplicity we will actually work with the germ of φ and will not concern ourselves with simple issues of domain.) The *base-point* of φ is $\pi(\varphi) := \varphi(0)$. The group of (germs at 0 of) local diffeomorphisms of $\mathbb{R}^n$ that fix 0 will be denoted by $\mathcal{H}$. A *section* of $\pi : \mathcal{J} \to M$ is a function $\sigma : M \to \mathcal{J}$ that to each $x \in M$ assigns an element $\sigma(x) \in \mathcal{J}$ whose base-point is x.

If $\Phi : G \times M \to M$ is a group action such that Φ_g is a diffeomorphism for each g, then a choice of section σ of $\mathcal{J}$ gives rise to a cocycle $c : G \times M \to \mathcal{H}$ by setting:

$$c(g,x) := \sigma(\Phi_g(x))^{-1} \circ \Phi_g \circ \sigma(x) \,. \tag{5}$$

If c is cohomologous to a ρ-simple cocycle for some linear representation $\rho : G \to GL(n,\mathbb{R}) \subset \mathcal{H}$ then we can find a function $u : M \to \mathcal{H}$ such that $c(g,x) = u(\Phi_g(x)) \circ \rho(g) \circ u(x)$. Setting $h(x) := \sigma(x) \circ u(x)^{-1}$ it follows that

$$\Phi_g \circ h(x) = h(\Phi_g(x)) \circ \rho(g) \,. \tag{6}$$

This means that, with respect to the moving frame of local coordinates given by h, the action reduces to a linear representation.

It should be clear that being able to linearize the action in the sense described here cannot always be done and, when it is available one gets a wealth of information about Φ. For example, if Φ preserves a Borel probability measure on a compact M, then a application of Poincaré recurrence shows that all the Lyapunov exponents of all elements $g \in G$ are the same as the exponents for the linear action of $\rho(g)$ on $\mathbb{R}^n$.

Klein model
locally homogeneous structure
invariant reduction
Cartan connections
orbit equivalence

Locally Homogeneous Structures. The linearization problem of the previous section can be generalized somewhat. Suppose that H is a Lie group and that M_0 is a smooth manifold with a transitive H-action. We will refer to the pair (M_0, H) as a *Klein model*. Fix $x_0 \in M_0$ and let J be the stabilizer of x_0 in H. Then M_0 can be identified with H/J while x_0 corresponds to the identity coset o.

A *locally homogeneous structure* with Klein model (M_0, H) on a manifold M may be defined as being given by a (say, smooth) atlas $\{(U_\alpha, \varphi_\alpha)\}$ on M such that $\varphi_\alpha : U_\alpha \to M_0$ is a diffeomorphism onto an open subset $V_\alpha \subset M_0$ and the changes of coordinates $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are the restrictions of an element of H.

Let $\mathcal{J}^H$ be the set of all germs of local parametrizations of M of the form $\varphi : V \subset M_0 \to M$, where V is a neighborhood of o and the inverse of φ belongs to the atlas defining the locally homogeneous structure. Let $\pi : \mathcal{J}^H \to M$ represent the base point projection: $\pi(\varphi) := \varphi(o)$. Notice that two elements $\varphi_1, \varphi_2 \in \mathcal{J}^H$ with the same base point must differ by some $h \in J$, that is $\varphi_2 = \varphi_1 \circ h$, so in this sense J acts (freely and transitively) on the fibers of $\mathcal{J}^H$. We view $\mathcal{J}^H$ as a subset of the $\mathcal{J}$ defined earlier, which will be regarded here as the set of all germs of local diffeomorphisms from a neighborhood of $o \in M_0$ into M.

The notion that a smooth action of a group G on M preserves a locally homogeneous structure (defined by $\mathcal{J}^H$) with Klein model (M_0, H) can now be described by the condition that the natural action of G on $\mathcal{J}$ actually sends $\mathcal{J}^H$ into itself. (That is, $\mathcal{J}^H$ is a *G-invariant reduction* of $\mathcal{J}$.) If σ is a section of $\mathcal{J}$ and c is the cocycle associated to σ (with values in the group of germs of local diffeomorphism of M_0 fixing o), then the existence of an invariant locally homogeneous structure corresponds to c being cohomologous to another cocycle taking values into J associated to a section of $\mathcal{J}^H$.

From the infinitesimal point of view, invariant locally homogeneous structures correspond to invariant (flat) *Cartan connections* associated to a Klein model. For more on this circle of ideas, see [6].

Orbit Equivalence. Yet another place where the notion of cocycles intervenes is in the context of orbit equivalence. The simple moral of the argument given next is this: to show that two orbit equivalent actions are isomorphic depends on showing that a certain cocycle over one action is cohomologous to a ρ-simple cocycle. We only explain below the key idea and in a very special setting.

Suppose that G is a connected, simply connected, Lie group that acts (by a, say, continuous action $\Phi : G \times M \to M$) on a manifold M and that H is a connected Lie group acting continuously and locally freely on another manifold N (by $\Psi : H \times N \to N$.) Suppose further that the two actions are *orbit equivalent*: there exists a (say, continuous) map $l : M \to N$ such that $l(\Phi_g(x))$ and $l(x)$ lie on the same H-orbit for each $g \in G$ and $x \in M$.

In other words, if H_x denotes the (discrete) stabilizer of $l(x)$, there exists a $\bar{c}(g,x) \in H/H_x$ such that $l(\Phi_g(x)) = \Psi_h(l(x))$ for any representative h of $\bar{c}(g,x)$. We denote by $c(g,x)$ the unique element of H such that $g \mapsto c(g,x)$ is a continuous lifting of $g \mapsto \bar{c}(g,x)$ with $c(e,x) = e$, where e denotes the identity of G and H. It follows that

$$l(\Phi_g(x)) = \Psi_{c(g,x)}(l(x)) \tag{7}$$

for each x and all $g \in G$. For a fixed x and $g_1, g_2 \in G$, it follows from 7 that

$$\begin{aligned} l(\Phi_{g_1 g_2}(x)) &= l(\Phi_{g_1}(\Phi_{g_2}(x))) \\ &= \Psi_{c(g_1,\Phi_{g_2}(x))}(l(\Phi_{g_2}(x))) \\ &= \Psi_{c(g_1,\Phi_{g_2}(x))}(\Psi_{c(g_2,x)} l(x)) \\ &= \Psi_{c(g_1,\Phi_{g_2}(x))c(g_2,x)}(l(x)) \,. \end{aligned}$$

Therefore

$$\Psi_{c(g_1 g_2,x)}(l(x)) = \Psi_{c(g_1,\Phi_{g_2}(x))c(g_2,x)}(l(x)) \tag{8}$$

for each x and all $g_1, g_2 \in G$. But G is connected, H_x is discrete and the correspondence $g \mapsto c(g,x)$ is continuous, so $c(g_1 g_2, x) = c(g_1, \Phi_{g_2}(x))c(g_2,x)$. The conclusion is that c is a cocycle over the G-action with values in H.

Let us say now that $H = G$ and that $\rho : G \to G$ is an inner automorphism of G, that is, ρ is given by conjugation by some $g_0 \in G$. Suppose that c is cohomologous to a ρ-simple cocycle. It follows easily from the definitions that there is a function $w : M \to G$ and a new map $\bar{l} : M \to N$, defined by $\bar{l}(x) := \Psi_{g_0 w(x)}(l(x))$ such that

$$\bar{l} \circ \Phi_g = \Psi_g \circ \bar{l} \,. \tag{9}$$

We have thus obtained a new orbit equivalence which, now, intertwines the two actions.

3 Cocycles and Principal Bundle Actions

The definition of cocycles over a group action that we gave earlier is quite adequate for measurable dynamics but it has some drawbacks when dealing with smooth dynamics, differential topology and differential geometry. We will now reformulate the concept in more geometric terms using the language of actions on principal bundles.

3.1 A More General Definition of Cocycle

We give now a notion of cocycle that generalizes the one given earlier, which will work better for us. The definition will be presented in the smooth category, although it will be evident how to modify it for topological, C^r, or simply measurable cocycles.

cocycle relative to an open cover

cohomologous cocycles

Let Φ be an action of a group G on a (topological or smooth) manifold M. We fix an open cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ of M. A cocycle over the action Φ taking values in H *relative to the open cover* $\mathcal{U}$ consists of the following data: for each $\alpha, \beta \in I$ and $g \in G$ for which $U_\alpha \cap \Phi_g^{-1}(U_\beta)$ is nonempty there exists a smooth map

$$c_{\alpha\beta g} := c_{\alpha\beta}(g, \cdot) : U_\alpha \cap \Phi_g^{-1}(U_\beta) \to H \tag{10}$$

(defined whenever their domains are nonempty) such that for all $g_1, g_2 \in G$, $\alpha, \beta, \gamma \in I$ and $x \in U_\alpha \cap \Phi_{g_2}^{-1}(U_\beta) \cap \Phi_{g_1 g_2}^{-1}(U_\gamma)$

$$c_{\gamma\alpha}(g_1 g_2, x) = c_{\gamma\beta}(g_1, \Phi_{g_2}(x)) c_{\beta\alpha}(g_2, x). \tag{11}$$

The functions $c_{\alpha\beta}$ are smooth functions of g and x, in the following sense: for every open subsets $U \subset M$ and $W \subset G$ such that $U \subset U_\alpha \cap \Phi_g^{-1}(U_\beta)$ for all $g \in W$, the function $c_{\alpha\beta}$ from $W \times U$ into H is smooth.

Suppose now that $\{c_{\alpha\beta}\}$ and $\{d_{\alpha\beta}\}$ are two cocycles into H over the same action and relative to the same covering $\{U_\alpha\}$. We say that they are *cohomologous* cocycles if we can find, for each α, (say, smooth) functions $h_\alpha : U_\alpha \to H$ such that, on the appropriate domains,

$$c_{\alpha\beta}(g, x) = h_\alpha(\Phi_g(x))^{-1} d_{\alpha\beta}(g, x) h_\beta(x) . \tag{12}$$

(If the two cocycles are relative to different open covers of M, we define the relation of being cohomologous by restricting all the maps to the common refinement of the two covers.)

3.2 From Cocycles to Bundle Actions and Back

From the viewpoint of differential topology, cocycles over group actions arise whenever an action by automorphisms of a fiber bundle is expressed in terms of a trivialization of an associated principal bundle. It is often convenient, however, to work directly on the principal bundle itself, rather than attempt to obtain a cocycle by violent means since, of course, not all bundles are topologically trivial.

We describe now a simple construction that produces, from a given cocycle over a group action another action of the same group on various kinds of fiber bundles. Returning to the notation used at the beginning of the notes, we suppose that $c : G \times M \to H$ is a cocycle over an action $\sigma : G \times M \to M$. We will regard H as a group of transformations of another manifold V. On the space $F(V) = M \times V$ (a trivial fiber bundle with fiber V and base M) we define an action Φ of G in the following way: for each $g \in G$ and $(x, u) \in F(V)$ write

$$\Phi_g(x, u) := (\sigma_g(x), c(g, x)u) . \tag{13}$$

The verification that Φ is indeed a group action rests on the cocycle property 2. So, for example, the random $\mathbb{Z}$-action on V considered at the beginning of the first section may be described as an ordinary $\mathbb{Z}$-action on the bigger space

$M \times V$, where M is the (compact) space of all bi-infinite sequences of 0s and 1s. By the natural projection $\pi : M \times V \to M$, the shift map, $\sigma : M \to M$, becomes a factor of the new action.

principal bundle

We can think of $M \times V$ and $M \times G$ as trivial bundles over M. The definition of a general cocycle relative to a covering is the exact information that we need in order to define an action of G by automorphisms of a *principal bundle* (with group H) covering σ.

We give here an illustration of the opposite process, that is, of obtaining a cocycle from an action on a fiber bundle. It will be supposed for simplicity that the bundle is trivial, although the idea readily generalizes. Suppose that M is a smooth n-dimensional manifold and that $\Phi : G \times M \to M$ is a smooth action of a Lie group G (possibly discrete). Then Φ gives rise in a natural way to an action, $\bar{\Phi}$, of G on the tangent bundle: $\bar{\Phi}_g(v) = d(\Phi_g)_x v$ for all $x \in M$ and $v \in T_x M$. A trivialization of TM can be obtained as follows. Suppose that we have chosen an identification of $T_x M$ with $\mathbb{R}^n$ for each $x \in M$. This is given by a choice, for each $x \in M$, of linear isomorphism $\eta(x) : \mathbb{R}^n \to T_x M$. By means of this choice the derivative map $d(\Phi_g)_x : T_x M \to T_{\Phi_g(x)} M$ can be represented by a linear automorphism of $\mathbb{R}^n$. Denoting this automorphism by $c(g, x) \in GL(n, \mathbb{R})$, we have:

$$c(g, x) := \eta(\Phi_g(x))^{-1} \circ d(\Phi_g)_x \circ \eta(x) . \tag{14}$$

The map $c : G \times M \to GL(n, \mathbb{R})$ is a cocycle over the action Φ taking values in the general linear group in dimension n.

A *section* of a principal bundle P over $U \subset M$ is a map $\sigma : U \to P$ such that $\sigma(x)$ is sent to x under the natural projection from P to M. Let $\bar{\Phi} : G \times P \to P$ be an action of G on a principal H-bundle P by automorphisms and suppose that $\mathcal{U} = \{U_\alpha\}$ is an open covering of M such that the restriction $P|_{U_\alpha} \to U_\alpha$ is a trivial bundle for each $\alpha \in I$. Choose a smooth section $\sigma_\alpha : U_\alpha \to P$ for each α. The choice of local sections gives rise to a cocycle $\{c_{\alpha\beta}\}$ relative to $\mathcal{U}$ defined by the equation

$$\bar{\Phi}_g(\sigma_\beta(x)) = \sigma_\alpha(\Phi_g(x)) c_{\alpha\beta}(g, x) . \tag{15}$$

The general correspondence between actions on principal bundles and cocycles is summarized in the proposition that follows. (The proof is a simple exercise and is left to the reader.)

Proposition 3.2.1 *Let $c = \{c_{\alpha\beta} : \alpha, \beta \in I\}$ be a smooth cocycle over an action $\Phi : G \times M \to M$, taking values in H, and relative to an open covering of M, $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$. Then there exists a unique (up to bundle isomorphism) smooth principal H-bundle P and an action of G on P by automorphisms covering Φ such that P admits a trivialization over $\mathcal{U}$ that gives rise (according to Equation 14) to a cocycle cohomologous to c.*

Some examples of principal bundles and G-actions defined on them are given next.

frame
frame bundle
frame of order r

Example 1. If H is a closed subgroup of a Lie group G, then the natural projection $p : G \to G/H$ makes G a principal H-bundle over G/H. Any subgroup of G acts on G and on G/H by left-translations, which are principal bundle automorphisms.

Example 2. Let Γ be a discrete subgroup of a Lie group, and suppose that $\rho : G \to L$ a smooth homomorphism into another Lie group L. Define $P = (G \times L)/\Gamma$, where Γ acts on the product by:

$$(g, l) \cdot \gamma = (g\gamma, \rho(\gamma)^{-1} l) . \tag{16}$$

Then, there is a well defined smooth projection $p : P \to M = G/\Gamma$ and a right-action of L on P making P a principal L-bundle over M. Any subgroup of G acts on P as follows: if $g' \in G$ and $[g, l] \in P$ is the element represented by (g, l), then $g'[g, l] := [g'g, l]$.

Example 3. The set of frames on the tangent bundle of a manifold, as defined earlier, is also a principal bundle, called the *frame bundle*. We recall that a *frame* at $x \in M$ is a linear isomorphism $\sigma : \mathbb{R}^n \to T_xM$. Notice that $GL(n, \mathbb{R})$ acts freely and transitively on the right on the set of all frames at x by

$$(\sigma, A) \mapsto \sigma \circ A ,$$

where $A \in GL(n, \mathbb{R})$ and for each $v \in \mathbb{R}^n$, $(\sigma \circ A)v = \sigma(Av)$. Here, Av denotes matrix multiplication of the column vector v by A. Let P be the set of all frames at all $x \in M$. It is a simple exercise to verify that P defines a smooth principal $GL(n, \mathbb{R})$-bundle, called the *frame bundle of order* 1, or simply the *frame bundle* of M. A group of diffeomorphisms of M naturally induces a left action on P by composition: $g\sigma := dg_x \circ \sigma$.

More generally, let $p : E \to M$ be the projection map of a vector bundle over M, whose standard fiber is an m-dimensional vector space V. We denote by $\mathcal{F}(E)$ the frame bundle of E, which is defined just as above for $E = TM$. Notice that a frame at $x \in M$ now corresponds to a linear isomorphism $\sigma : V \to E_x$ and that a group action on E by automorphisms naturally induces an action on $\mathcal{F}(E)$.

Example 4. A *frame of order* r at $x \in M$ is an equivalence class of smooth parametrizations at x under the equivalence relation that identifies two parametrizations $\varphi : U_0 \to U$ and $\psi : V_0 \to V$ if for each smooth function $f : M \to \mathbb{R}$ all the partial derivatives at 0 of $f \circ \varphi$ and $f \circ \psi$ up to order r coincide. (A (smooth) parametrization of an open subset $U \subset M$ is a (smooth) diffeomorphism from an open subset of $\mathbb{R}^n$ onto U. We say that $\varphi : U_0 \to U$ is a parametrization at x if $0 \in U_0$, $x \in U$ and $\varphi(0) = x$.) In particular, a frame of order 1 at x is a linear isomorphism from $\mathbb{R}^n$ onto the tangent space T_xM. In the general case, the equivalence class represented by a parametrization φ will be denoted $(j^r\varphi)_0$ – the r-th jet of φ at 0.

frame bundle of order r

The collection of all frames over points of M forms in a natural way a smooth manifold, which will be called the r-th order *frame bundle of* M and will be denoted $F^r(M)$. This is indeed a locally trivial fiber bundle over M and the bundle map $\pi : F^r(M) \to M$ is the obvious base point projection that to each $(j^r\varphi)_0$ associates $\varphi(0)$.

Having fixed a frame $\xi = (j^r\varphi)_0$ at x, any other frame of order r at the same point is given by ξg, where $g = (j^r f)_0$ is the r-jet of a diffeomorphism f from a neighborhood of 0 into another neighborhood of 0 such that $f(0) = 0$. By definition,

$$\xi g := j^r(\varphi \circ f)_0 \,.$$

The collection of all r-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ fixing 0 forms a Lie group, denoted here $G^r = G^r(n, \mathbb{R})$. Notice that G^1 is the general linear group $GL(n, \mathbb{R})$. It can be shown that G^r is in a natural way a linear real algebraic group. (G^r can be regarded as a subgroup of $GL(V^{r-1})$ of all invertible linear transformation of the vector space of $r-1$ jets at 0 of smooth vector fields on $\mathbb{R}^n$.)

The map $F^r(M) \times G^r \to F^r(M)$ given by $(\xi, g) \mapsto \xi g$ is a smooth group action that sends each fiber of $F^r(M)$ onto itself. It is clear, furthermore, that the action is transitive on each fiber. With this action, $F^r(M)$ becomes a principal bundle. A smooth parametrization of an open subset $U \subset M$ can be used to trivialize $F^r(M)$ above U, making $\pi^{-1}(U) \subset F^r(M)$ isomorphic to the trivial bundle $U \times G^r$.

Any smooth action on M naturally induces an action by automorphisms of $F^r(M)$.

3.3 $\Gamma(E)$-Valued Cocycles

There is another sort of cocycle that often arises in dynamics (for example, in problems about existence of fixed points; one such problem will be seen later concerning invariant connections), which we want to relate to the present discussion.

Suppose that $\pi_E : E \to M$ is a vector bundle over a manifold M. It may be assumed that E is an associated vector bundle to a principal H-bundle $\pi_P : P \to M$. In other words, we may write $E = (P \times V)/H$, where V is a finite dimensional vector space on which H acts via a linear representation $\eta : H \to GL(V)$. The quotient consists of orbits of the H-action on $P \times V$ defined by:

$$(p, v)h := (ph, \eta(h)^{-1}v) \,. \tag{17}$$

If a group G acts on P by automorphisms, it also naturally acts on E by operating on the first factor of the product.

From P and E we define another principal bundle:

$$\pi : P \times_M E = \{(p, \alpha) \in P \times E : \pi_P(p) = \pi_E(\alpha)\} \to M \tag{18}$$

affine cocycle
pseudo-Riemannian metric

whose structure group is $A := H \ltimes_\eta V$. By definition, the multiplication in A is given by

$$(h,u)(h',u') = (hh', \eta(h)u' + u) \tag{19}$$

while the right-action of A on $P \times_M E$ is defined by

$$(p,\alpha)(h,u) = (ph, \alpha - pu)\,. \tag{20}$$

(An element $p \in P$ in the fiber of $x \in M$ can be regarded as a linear isomorphism from V onto the fiber $E_x := \pi_E^{-1}(x)$, thus the notation pu.)

Let now $\theta : G \to \Gamma(E)$, where $\Gamma(E)$ denotes a space of sections of E (with some specified degree of regularity, say continuous sections, although regularity is not a concern at this point). Then it is a simple exercise to check that

$$g(p,\alpha) := (gp, g\alpha + \theta(g)(\pi_P(gp))) \tag{21}$$

will define a G-action by principal bundle automorphisms (of $P \times_M E$) if and only if θ satisfies the following identity:

$$\theta(g_1 g_2) = \theta(g_1) + g_{1*}\theta(g_2) \tag{22}$$

where $(g_*\alpha)(x) := g\alpha(g^{-1}x)$, with $\alpha = \theta(g_2)$.

We will call θ, satisfying 22, an *affine cocycle*. If there is a section α of E such that $\theta(g) = \alpha - g_*\alpha$ for all $g \in G$, we will say that θ is an *affine coboundary.*

The next proposition is an elementary consequence of the definitions.

Proposition 3.3.1 *The bundle $P \times_M E$ with a G-action defined by an affine cocycle θ admits a G-invariant reduction Q with group H if and only if θ is a coboundary.*

The quotient of the linear space of affine cocycles by the subspace of affine coboundaries (with values in $\Gamma(E)$) will be written $H^1_s(G,E)$, where s will indicated the degree of regularity of the sections of E.

3.4 Invariant Geometric Structures

A smooth action of G on a smooth manifold M may preserve some geometric structure like, for example, a pseudo-Riemannian metric on M. We use this example to explain how the presence of some kind of geometry on M, with respect to which G operates as a group of symmetries, is reflected by the properties of cocycles over the action.

Pseudo-Riemannian Metrics. A pseudo-Riemannian metric on M is defined by the choice at each $x \in M$ of a, not necessarily positive definite, inner product $b_x = \langle \cdot,\cdot \rangle_x$ on T_xM. (The metric is smooth if the field $x \mapsto b_x$ of bilinear symmetric forms is smooth.) A smooth action Φ of G on M is said to preserve the metric if for each $g \in G$, $x \in M$, and $u, v \in T_xM$

$$\langle d(\Phi_g)_x u, d(\Phi_g)_x v\rangle_{\Phi_g(x)} = \langle u, v\rangle_x \,. \tag{23}$$

An *orthonormal frame* at $x \in M$ (for the pseudo-Riemannian metric b) is a linear isomorphism $\eta : \mathbb{R}^n \to T_xM$ that maps the standard basis of $\mathbb{R}^n$ into an orthonormal basis of T_xM relative to b. We recall that an orthonormal basis, $\{u_1, \ldots, u_p, u_{p+1}, \ldots, u_n\}$, at T_xM is a basis such that

$$\langle u_i, u_j\rangle_x = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } 1 \leq i = j \leq p \\ -1 & \text{if } p+1 \leq i = j \leq n\,. \end{cases}$$

The integer $s = 2p - n$ is the *signature* of b. The change of basis matrix between two orthonormal basis for b is an element of the pseudo-orthogonal group $O(p, n-p)$.

If, now, $x \mapsto \eta(x)$ is a (say, measurable) choice of a b-orthonormal frame at T_xM for each x, then it follows from (10) and (11) that the associated cocycle, $c : G \times M \to GL(n, \mathbb{R})$, for a b-preserving action on the tangent bundle of M (induced from a smooth action Φ on M) actually takes values in the subgroup $O(p, n-p)$.

It can happen that a (generalized) cocycle over Φ (a smooth action of G on M) taking values in H is cohomologous to a cocycle into a smaller group, $L \subset H$. This means that we would have obtained a G-invariant reduction of the original H-bundle (associated to the first cocycle). Such an invariant reduction is interpreted as additional structure preserved by the action. For example, any smooth action on M produces an action on the frame bundle $F(M)$ (therefore, it gives rise to $GL(n, \mathbb{R})$-cocycles). If a pseudo-Riemannian metric is preserved by the action, we obtain a reduction of the frame bundle to an $O(p, n-p)$-principal bundle. Equivalently, the original cocycle is cohomologous to a cocycle into $O(p, n-p)$.

Invariant Volume Forms. If Ω is a smooth volume form (a non-vanishing n-form) on an n-dimensional manifold M and the G-action on M is smooth, then for each $g \in G$ we can define a smooth function $\theta(g)$ on M such that

$$(g^{-1})^*\Omega = e^{\theta(g)}\Omega\,. \tag{24}$$

It can be easily checked that θ is an affine cocycle and that it is a (smooth) coboundary precisely when there exists a (smooth) invariant volume form. In fact, if $\theta(g) = \alpha - \alpha \circ g^{-1}$ for a smooth function α, then $e^{\alpha}\Omega$ is an invariant volume form.

invariant connection

Invariant Connections. A more interesting example, but similar to the previous one, is the following. Suppose that ∇ is an affine connection on M (say, continuous). The transformation (push-forward) of ∇ under a diffeomorphism f will be denoted by $f \cdot \nabla$. Then for each $g \in G$, $\theta(g) := \nabla - g \cdot \nabla$ is a continuous affine cocycle with values into the space of continuous sections of $T^*M \otimes T^*M \otimes TM$. If there is a continuous section α of this vector bundle such that $\theta(g) = \alpha - g_*\alpha$ then $\nabla' := \nabla - \alpha$ is a continuous G-invariant connection.

3.5 Measurable Geometric Structures.

We describe here an example of a (measurable) invariant geometric structure that exists whenever G is $\mathbb{R}$ or $\mathbb{Z}$. (Or, more generally, the G-action is amenable.)

Let P be a principal H-bundle over M, V an H-space, and T a group of automorphisms of P such that $T \times P \to P$ is a continuous action. A T-invariant geometric structure of type V may be defined as a T-invariant section of the associated bundle $P \times_H V$. We have already seen for pseudo-Riemannian metrics how invariant structures yield invariant reductions of P. As another example, a measurable field of m-dimensional planes on the n-dimensional manifold M can be described as a measurable section η of the associated bundle $F(M) \times_{GL(n,\mathbb{R})} V$, where V is the Grassmannian variety of m-planes in $\mathbb{R}^n$, and $\eta(x)$ is interpreted as an m-dimensional subspace of T_xM at each $x \in M$. If such a measurable plane field is invariant under T and the action of T on M is ergodic with respect to some invariant probability measure, then we obtain a measurable T-invariant L-reduction of P where L is the subgroup of $GL(n, \mathbb{R})$ that stabilizes $\mathbb{R}^m$.

A "probabilistic" generalization of this notion of geometric structure, which is implicit in Zimmer's proof of the cocycle super-rigidity theorem ([11]), is the following. Suppose that instead of, say, a field of m-dimensional planes that are exactly specified at each $x \in M$ we have a "field of probability distributions" so that at each x, one is given a probability measure μ_x on the Grassmannian variety V_x of m-dimensional subspaces of T_xM; that is, the m-plane at x is only specified "up to probability μ_x". The field of probabilities is said to be invariant if the following holds: let τ be an element of T and let the induced map on $P \times_H V$ also be denoted by τ, so that for each $x \in M$, $\tau : V_x \to V_{\tau x}$. Then the invariance condition is that $\tau_*\mu_x = \mu_{\tau x}$ for each x and τ. When μ_x is the Dirac measure supported on a single point of V_x we recover the "deterministic" notion of a plane field.

More precisely, let V be a compact metric space and $\mathcal{M}(V)_1$ the space of Borel probability measures on V. $\mathcal{M}(V)_1$ is a compact convex metrizable space and the group of homeomorphisms of V acts on $\mathcal{M}(V)_1$ via a continuous action by homeomorphisms. Form the associated bundle $P \times_H \mathcal{M}(V)_1$. Then a Borel measurable section of this bundle is our field of probabilities, $x \mapsto \mu_x$, where each μ_x is a probability measure on the fiber of $P \times_H V$ above x.

Cartan involution
reductive group

Such a section can also be represented by a measurable H-equivariant map $\mathcal{G} : P \to \mathcal{M}(V)_1$.

A reason for considering such spaces of measures as fibers of a bundle is that we can use weak compactness to obtain the existence of invariant sections, when the acting group is amenable. The next proposition is a reformulation (proved in [5]) of related statements contained in [11].

Proposition 3.5.1 *Let P be a topological principal H-bundle over a manifold M with projection map p. Suppose that τ is a bundle automorphism of P and that the group T generated by τ induces an ergodic action on M with respect to a T-invariant probability measure μ. Suppose that H is a real algebraic group and let B be a real algebraic subgroup of H such that the quotient $V := H/B$ is compact. Form the associated bundle $P_V := P \times_H V$, where H acts on V by left-translations. Then there exists a T-invariant measurable L-reduction of $P|_A$, where A is a T-invariant measurable subset of full μ-measure and $L = H_{\nu_0}$ is the isotropy subgroup of a measure $\nu_0 \in \mathcal{M}(V)_1$.*

4 Semisimple Lie Groups, in a Hurry

This section collects some basic facts about semisimple Lie groups and Lie algebras that will be needed later on.

4.1 Definitions and Examples

We only consider here linear Lie groups, that is, (real) subgroups of $GL(n, \mathbb{C})$. Let A^* denote the complex conjugate transpose of A. The *Cartan involution* of $GL(n, \mathbb{C})$ is the homomorphism

$$\Theta : A \mapsto (A^*)^{-1} .$$

The Cartan involution of $\mathfrak{gl}(n, \mathbb{C})$ (the Lie algebra of $GL(n, \mathbb{C})$) is the Lie algebra isomorphism induced from Θ, and is given by $\theta : X \mapsto -X^*$. Θ is indeed a group homomorphism and an involution, i.e. Θ^2 is the identity map, as one can easily check.

Let G be a connected Lie subgroup of $GL(n, \mathbb{C})$. We say that G is a *reductive* group if it is conjugate to a subgroup that is stable under the Cartan involution Θ. In other words, G is reductive if there is $g \in GL(n, \mathbb{C})$ such that gGg^{-1} is mapped into itself by Θ. A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C})$ is reductive if it is conjugate by an element in $GL(n, \mathbb{C})$ to a θ-stable subalgebra. In particular, G is reductive if and only if $\mathfrak{g}$ is.

We recall that the center of a group G is the subgroup

$$Z(G) = \{a \in G \mid ag = ga \text{ for all } g \in G\} .$$

semisimple Lie algebra
semisimple Lie group
simple Lie algebra
split Cartan subalgebra
real rank

The center is clearly a normal subgroup of G. The center of a Lie algebra $\mathfrak{g}$ is the subalgebra

$$\mathfrak{z}(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{g}\}$$

and is an ideal of $\mathfrak{g}$. (A subalgebra $\mathfrak{n} \subset \mathfrak{g}$ is an ideal if $[X, Y] \in \mathfrak{n}$ for all $X \in \mathfrak{n}$ and all $Y \in \mathfrak{g}$.)

A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{C})$ is *semisimple* if it is reductive and has trivial center. $G \subset GL(n, \mathbb{C})$ is a *semisimple Lie group* if its Lie algebra is semisimple.

A Lie algebra is said to be *simple* if its only ideals are $\{0\}$ and itself.

4.2 Real Rank

Since θ is an involution, i.e. $\theta^2 = \mathrm{id}$, its only eigenvalues are 1 and -1. We define subspaces $\mathfrak{k}$ and $\mathfrak{p}$ of the θ-stable Lie algebra $\mathfrak{g}$ as follows:

$$\begin{aligned} \mathfrak{k} &:= \{X \in \mathfrak{g} \mid \theta(X) = X\} \\ \mathfrak{p} &:= \{X \in \mathfrak{g} \mid \theta(X) = -X\} . \end{aligned}$$

Since θ is a Lie algebra automorphism $\mathfrak{k}$ is a Lie subalgebra. Let K be the subgroup of G fixed by Θ. Its Lie algebra is clearly $\mathfrak{k}$.

As a vector space, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Introduce in $\mathfrak{g}$ an inner product by

$$\langle X, Y \rangle = -\mathrm{Re}(\mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(\theta Y)))$$

where $\mathrm{ad}(X)$ is the linear map on $\mathfrak{g}$ defined by $\mathrm{ad}(X)Z = [X, Z]$. For each $X \in \mathfrak{p}$, the operator $\mathrm{ad}(X)$ on $\mathfrak{g}$ is self-adjoint with respect to the given inner product. Therefore $\mathrm{ad}(X)$ is diagonalizable with real eigenvalues. Let $\mathfrak{a}$ be a maximal Abelian algebra in $\mathfrak{p}$. More precisely, $\mathfrak{a}$ is Abelian and is not properly contained in a subspace of $\mathfrak{p}$ consisting of commuting elements. The operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$, commute since

$$0 = \mathrm{ad}([X, Y]) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)$$

for $X, Y \in \mathfrak{a}$. Therefore, it is possible to find a basis for $\mathfrak{g}$ which simultaneously diagonalizes all the operators $\mathrm{ad}(X)$, $X \in \mathfrak{a}$. The subalgebra $\mathfrak{a}$ will be called an $\mathbb{R}$-*split Cartan subalgebra* of $\mathfrak{g}$. A more descriptive name is "maximal Abelian $\mathbb{R}$-diagonalizable subalgebra." The dimension of $\mathfrak{a}$ is called the *real rank* of $\mathfrak{g}$. This definition seems to depend on the choice of $\mathfrak{a}$ in $\mathfrak{p}$. It turns out, however, that any two such subalgebras are conjugate by an element of K, so that their dimensions are the same.

It is not hard to show, for example, that the real rank of $SL(n, \mathbb{R})$ is $n-1$.

4.3 Root Space Decomposition

root
root space
root space decomposition

Let $\mathfrak{a}$ be an $\mathbb{R}$-split Cartan subalgebra contained in $\mathfrak{p}$. Denote by $\mathfrak{a}^*$ the space of real linear functionals on $\mathfrak{a}$. We now define

$$\mathfrak{g}_\lambda := \{X \in \mathfrak{g} \mid [H, X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \in \mathfrak{a}^*$ is nonzero and $\mathfrak{g}_\lambda$ is nonzero, we say that λ is a *root* of $(\mathfrak{a}, \mathfrak{g})$ (or a *restricted root* of $\mathfrak{g}$), with associated root space $\mathfrak{g}_\lambda$. The set of all such roots is denoted $\Phi(\mathfrak{a}, \mathfrak{g})$. We denote by $\mathfrak{g}_0$ the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, i.e. $\mathfrak{g}_0$ is the subspace of all X in $\mathfrak{g}$ such that $[X, H] = 0$ for all $H \in \mathfrak{a}$. Therefore, we have the direct sum decomposition of vector spaces:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda \in \Phi(\mathfrak{a}, \mathfrak{g})} \mathfrak{g}_\lambda$$

called the *restricted root space decomposition* of $\mathfrak{g}$.

Example: Root Spaces for $SL(n, \mathbb{F})$. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$, where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. In this case $\mathfrak{k}$ is the subalgebra of skew-Hermitian $n \times n$ matrices (skew-symmetric if $\mathbb{F} = \mathbb{R}$) and $\mathfrak{p}$ is the subspace consisting of Hermitian $n \times n$ matrices of trace 0. Denote by $\mathfrak{a}$ the Abelian algebra consisting of real diagonal matrices of trace 0. Then $\mathfrak{a} \subset \mathfrak{p}$, and by a simple computation we see that the subalgebra consisting of all matrices in $\mathfrak{p}$ that commute with each element of $\mathfrak{a}$ is $\mathfrak{a}$ itself. Therefore, $\mathfrak{a}$ is an $\mathbb{R}$-split Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{F})$ and the real rank of $\mathfrak{sl}(n, \mathbb{F})$ is $n - 1$.

For each i, $1 \leq i \leq n$, define $f_i \in \mathfrak{a}^*$ by $f_i(\operatorname{diag}[a_1, \ldots, a_n]) = a_i$ and set $\alpha_{ij} := f_j - f_i$. Define $\mathfrak{g}_{ij} := \mathbb{F}E_{ij}$, where E_{ij} is the matrix with 1 at the ij-entry and 0 at the other positions. Notice that $\dim \mathfrak{g}_{ij} = 1, 2, 4$ for $\mathbb{R}, \mathbb{C}, \mathbb{H}$. One can easily check that

$$\mathfrak{sl}(n, \mathbb{F}) = \mathfrak{g}_0 \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}$$

where $\mathfrak{g}_0$ is the subalgebra of all diagonal matrices of trace 0. We can write $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is trivial if $\mathbb{F} = \mathbb{R}$, $\mathfrak{m}$ is the subalgebra of all diagonal matrices with imaginary entries for $\mathbb{F} = \mathbb{C}$, and $\mathfrak{m}$ is the direct sum of n copies of $\mathfrak{su}(2)$ if $\mathbb{F} = \mathbb{H}$. (Notice that $\mathfrak{su}(2)$ is isomorphic to the algebra of imaginary quaternions.)

4.4 The Rank-at-Least-Two Assumption

We now introduce the assumption that $\dim \mathfrak{a} \geq 2$; in other words, G has real rank at least 2. For each $\alpha \in \Phi(\mathfrak{a}, \mathfrak{g})$, let $H_\alpha \in \mathfrak{a}$ be the dual vector to α. The orthogonal complement in $\mathfrak{a}$ (relative to the inner product $\langle \cdot, \cdot \rangle$) of the line

$\mathbb{R}H_\alpha$ is the hyperplane denoted $H_\alpha^\perp$. Notice that $H_\alpha^\perp$ is nonzero by the rank assumption. Since $\alpha(H) = \langle H_\alpha, H\rangle$, we have $H_\alpha^\perp = \ker(\alpha)$.

The centralizer of $H_\alpha^\perp$ in G will be denoted

$$Z_\alpha = \{g \in G \mid gHg^{-1} = H, \text{ for all } H \in H_\alpha^\perp\}\,.$$

The next proposition will be fundamental in the proof of the main result of these notes.

Proposition 4.4.1 *Suppose that G has real rank at least 2. Then each $g \in G$ can be written as a product $g = g_1 g_2 \cdots g_l$ where, for each i, $1 \le i \le l$, $g_i \in Z_\alpha$ for some $\alpha \in \Phi(\mathfrak{a}, \mathfrak{g})$.*

Proof. We only explain the proof in the special case $G = SL(3,\mathbb{R})$. (See [3] for the general case.) Let N^- (respectively, N) denote the group of upper (respectively, lower) triangular matrices in G with diagonal entries equal to 1, and let A be the Abelian group of diagonal matrices of determinant 1 and positive diagonal entries. This is a nilpotent group. The set AN, comprising the elements of G of the form hg for $h \in A$ and $g \in N$ is also a (solvable) group. We claim that the image of the multiplication map

$$\begin{aligned} m : N^- \times AN &\to SL(3,\mathbb{R}) \\ (n,h) &\to nh \end{aligned}$$

is an open dense subset of G. (This can be easily proved by applying the row reduction method. If $g = (a_{ij})$, the condition for g to split as a product nh, that is, for g to be in the image of m, is that $a_{33} \neq 0$ and $a_{33}a_{22} - a_{32}a_{23} \neq 0$.) We denote by $\mathcal{W}$ the image of the map m.

Notice that every $g \in G$ can be written as a product $g = w_1 w_2$, for $w_i \in \mathcal{W}$. In fact, since $\mathcal{W}$ is open and dense, the same is true about $g\mathcal{W}^{-1}$ and $\mathcal{W} \cap g\mathcal{W}^{-1}$. The latter is, therefore, nonempty so that we can find $w_1 \in \mathcal{W}$ such that $w_1 = g w_2^{-1}$ for some $w_2 \in \mathcal{W}$.

Consider the one-dimensional groups $U_i^\pm$, $i = 1,2,3$, defined as follows:

$$U_1^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$
$$U_2^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$$
$$U_3^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

The groups U_i^- are defined by taking the transpose (or, rather, the image under Θ) of the U_i^+. It is now an easy calculation to check that the two maps

$$\begin{aligned} U_1^\pm \times U_2^\pm \times U_3^\pm &\to N^\pm \\ (u_1, u_2, u_3) &\mapsto u_1 u_2 u_3 \end{aligned}$$

are diffeomorphisms. We label the roots as follows:

$$\alpha_1 = f_2 - f_3,\ \alpha_2 = f_1 - f_3,\ \alpha_3 = f_1 - f_2.$$

The kernel of α_i is the space $H_i^\perp := H_{\alpha_i}^\perp$. A simple matrix computation shows that

$$\begin{aligned} H_1^\perp &= \left\{ \begin{pmatrix} -2a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{R} \right\} \\ H_2^\perp &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -2a & 0 \\ 0 & 0 & a \end{pmatrix} : a \in \mathbb{R} \right\} \\ H_3^\perp &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a \in \mathbb{R} \right\} . \end{aligned}$$

Given what we have shown so far, in order to prove the proposition (in this special case) it suffices to check that the union of the centralizers $Z_i := Z_{\alpha_i}$ of $H_i^\perp$, $i = 1, 2, 3$, contains the groups A and $U_i^\pm$. A glance at the expressions given next makes this clear:

$$\begin{aligned} Z_1 &= \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in SL(3, \mathbb{R}) \right\} \\ Z_2 &= \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix} \in SL(3, \mathbb{R}) \right\} \\ Z_3 &= \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in SL(3, \mathbb{R}) \right\} . \end{aligned}$$

□

5 All the Ergodic Theory we Need

We review here some general facts about the ergodic theory of actions of semisimple groups.

reduction lemma

5.1 The Reduction Lemma

Some of the missing details in the present section can be found fully spelled out in [3]. For the reader who wishes to skip the section on a first reading, the whole point can be summarized thus: We all know that if a measurable function (into a nice Borel space) is invariant under an ergodic action, then the function is constant almost everywhere. More generally, we could be presented with a not necessarily invariant, but G-equivariant, function $\varphi : M \to V$ from a measure G-space M with finite, invariant, ergodic measure, into a real algebraic variety V, upon which the same G acts via an algebraic action. Then, the conclusion that φ is constant almost everywhere still holds. A similar fact holds after replacing 'measurable' with 'continuous' and 'ergodic' with 'topologically transitive'. (See Corollary 5.1.3.)

Now, on with the details. The next proposition shows that if a geometric A-structure ω admits a group of isometries that acts on M topologically transitively, then ω is an L-structure, at least over an open and dense subset of M.

Proposition 5.1.1 (Reduction Lemma) *Let V be a real algebraic H-space and P a principal H-bundle over a manifold M. Let $\mathcal{G} : P \to V$ be a C^r, $r \geq 0$, H-equivariant map and suppose that a group G of automorphisms of P acts topologically transitively on M. Suppose moreover that $\mathcal{G}$ is G-invariant. Then there exists an open and dense G-invariant subset U of M such that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit, $H \cdot v_0 \subset V$, for some $v_0 \in V$. The set $\mathcal{G}^{-1}(v_0) \subset P$ is a C^r G-invariant L-reduction of P, where $L \subset H$ is the isotropy subgroup of v_0. If $H \cdot v_0$ is a closed subset of V, then $U = M$.*

Proof. Suppose that $x_0 \in M$ has a dense G-orbit in M and let $\xi_0 \in P_x$ be any point in the fiber of P above x_0. Set $v_0 = \mathcal{G}(\xi_0)$ and denote by W the closure of the H-orbit of v_0 in V. Since the G-orbit of x_0 is dense in M, the $G \times H$-orbit of ξ_0 is also dense in P, and maps into $H \cdot v_0$. Therefore $\mathcal{G}$ maps P into W. But orbits of an algebraic action are locally closed, embedded submanifolds (see Corollary 4.9.3 of [3]). Therefore, $H \cdot v_0$ is open in W, and $\mathcal{G}^{-1}(H \cdot v_0)$ is an open and dense subset of P. This set is saturated by H-orbits since $\mathcal{G}$ is H-equivariant, hence it is of the form $P|_U$ for some open and dense subset $U \subset M$. Moreover, U is G-invariant since $\mathcal{G}$ is itself G-invariant. If $H \cdot v_0$ is closed in V, then $W = H \cdot v_0$, so that $U = M$. Once we know that $\mathcal{G}$ maps into a single H-orbit, it follows that we obtain a reduction as claimed. □

Proposition 5.1.2 *Let P be a principal H-bundle over M, where H is a real algebraic group, and let V be a real algebraic H-space. Let S be a real algebraic 1-parameter group also acting algebraically on V such that the H and S actions commute. Let T be a 1-parameter group of bundle automorphisms of P (hence the T-action commutes with the H-action on P), and let $\mathcal{G}$: $P|_U \to V$ be an $H \times T$-equivariant map defined over some subset $U \subset M$,*

Borel density theorem
Moore's ergodicity theorem

where equivariance means in this case that there is a smooth homomorphism $\rho : T \to S$ such that $\mathcal{G}(l\xi h) = h^{-1}\rho(l)\mathcal{G}(\xi)$ for each $l \in T$, $h \in H$ and $\xi \in P$. Suppose moreover that T acts topologically transitively on M, U is a nonempty T-invariant open set in M, and $\mathcal{G}$ is continuous. Then, there exists an open dense T-invariant subset $U' \subset U$ such that $\mathcal{G}|_{P_{U'}}$ takes values in a single H-orbit in V.

Proof. The main difference between this more complicated statement and the reduction lemma is that here $\mathcal{G}$ is not necessarily T-invariant but only equivariant. The proof makes use of a theorem of Rosenlicht. The reader is referred to [3]. □

Corollary 5.1.3 *Let V be a real algebraic S-space, where S is a real algebraic 1-parameter group. Let T be a 1-parameter group of homeomorphisms of a topological space M acting topologically transitively. Suppose that $\phi : U \to V$ is a continuous map defined on an open dense T-invariant subset $U \subset M$ and T-equivariant, i.e., there is a smooth homomorphism $\rho : T \to S$ such that $\varphi(lx) = \rho(l)\varphi(x)$ for each $l \in T$ and each $x \in U$. Then, ϕ is constant and its value is an S-fixed point. The same result, with the obvious modifications holds for ergodic actions.*

Proof. Set $H = S$ and $P = M \times S$. T acts on P by $l(x, s') = (lx, s')$ and S acts on the right: $(x, s')s = (x, s's)$. Define $\mathcal{G}(x, s) := s^{-1}\phi(x)$. Since S is Abelian, the conditions of the proposition are satisfied here, and the claim follows. □

It was first observed by Furstenberg, I believe, that results such as Corollary 5.1.3 generalize the next fundamental fact.

Theorem 5.1.4 (Borel's Density Theorem) *Let G be a connected real algebraic group generated by algebraic one-parameter subgroups. (For example, a noncompact connected simple real algebraic group such as $SL(n, \mathbb{R})$.) Then any lattice Γ of G is Zariski dense in G.*

Proof. Let L be the Zariski closure of Γ in G and apply Corollary 5.1.3 to the identity map $G/L \to G/L$ to conclude that it is a constant map. Therefore G/L is a single point and, consequently, $G = L$. □

5.2 Moore's Ergodicity Theorem

We state here a fundamental fact concerning the ergodic theory of measure preserving actions of semisimple groups. The reader can find a proof in many places, such as [3] for example.

irreducible G-space
algebraic hull

Let X be a G-space with a finite invariant measure. X is called *irreducible* if every normal subgroup of G not contained in the center acts ergodically on X. The next theorem is known as Moore's ergodicity theorem.

Theorem 5.2.1 *Suppose that G is a semisimple Lie group with finite center and no compact simple factors (that is, no compact nonabelian normal subgroup), and that X is an irreducible G-space with finite G-invariant measure. If H is a closed noncompact subgroup of G, then H also acts ergodically on X.*

6 The Algebraic Hull

Algebraic hulls are a useful abstraction for working in a uniform way with the collection of all invariant geometric structures (of algebraic type) possessed by a given action. The definition and basic properties are explained in this section.

6.1 Definition and Existence

The next proposition is a smooth version of Zimmer's result concerning the existence and uniqueness of measurable algebraic hulls [11].

Proposition 6.1.1 *Let H be a real algebraic group and let P be a principal H-bundle on which G acts by bundle automorphisms over a smooth, topologically transitive G-action on M. Then:*

1. *There exists a real algebraic subgroup $L \subset H$ and a smooth G-invariant L-reduction $Q \subset P|_U$, over a G-invariant dense set $U \subset M$, such that Q is minimal in the following sense: Q does not admit a smooth G-invariant L'-reduction on $P|_{U'}$, for some open and dense G-invariant U' and a proper real algebraic subgroup L' of L.*
2. *If Q_1 and Q_2 are G-invariant reductions with groups L_1 and L_2, respectively, satisfying the above minimality property, then there is an $h \in H$ such that $L' = hLh^{-1}$ and $Q_2 = Q_1 h^{-1}$.*
3. *Any G-invariant smooth L'-reduction of $P|_U$, for real algebraic L' and some invariant open dense U, contains a G-invariant L''-reduction of similar kind, where L'' is a conjugate in H of the minimal L obtained in item* 1.

Proof. Suppose that we have a nested sequence of invariant reductions $Q_1 \supset Q_2 \supset \cdots$ with groups $L_1 \supset L_2 \supset \cdots$. The groups L_i form a descending chain of real algebraic groups. By the descending chain condition for algebraic groups, the sequence must stabilize at a finite level, so that a minimal reduction must exist. The uniqueness claimed in item 2 can be seen as follows. A G-invariant L_i-reduction, Q_i, yields a G-invariant H-equivariant map

$$\mathcal{G}_i : P \to H/L_i \,. \tag{25}$$

Taking the product $\mathcal{G}_1 \times \mathcal{G}_2$, we obtain a G-invariant, H-equivariant map

$$\mathcal{G} : P \to H/L_1 \times H/L_2 \,. \tag{26}$$

The right-hand side in 26 is an H-space with the natural product action. Applying the reduction lemma to $\mathcal{G}$ we conclude that $\mathcal{G}$ maps $P|_U$ onto a single H-orbit in $H/L_1 \times H/L_2$, where U is a G-invariant open and dense subset of M. We denote that orbit by $H \cdot (h_1L_1, h_2L_2)$. The isotropy group of (h_1L_1, h_2L_2) is

$$L = \{h \in H \mid hh_1L_1 = h_1L_1, hh_2L_2 = h_2L_2\} \tag{27}$$

and we have a G-invariant measurable L-reduction Q of P. Notice that $L \subset h_1L_1h_1^{-1} \cap h_2L_2h_2^{-1}$. L cannot be a proper subgroup of $h_iL_ih_i^{-1}$ since, otherwise, Qh_i would define a proper reduction of Q_i, contradicting the minimality of Q_i. Therefore, $Qh_i = Q_i$, $i = 1, 2$, proving 2. A similar argument also shows 3. □

The conjugacy class of the group L obtained above is called the smooth *algebraic hull* of the G-action on P. By abuse of language, we sometimes call L itself the algebraic hull. Similarly one defines the measurable and C^r hulls.

An Example. Let Γ be a discrete subgroup of a Lie group G and $\rho : G \to L$ a smooth homomorphism into a real algebraic group L. Form the principal L-bundle $p : P = (G \times L)/\Gamma \to M = G/\Gamma$. If $\rho(\Gamma)$ is Zariski-dense in L, then it is not difficult to show that the C^r algebraic hull of the G-action on P is L. (If $g \in G$ and (g_0, l_0) represents an element $\xi \in P$, then $g\xi$ is the element represented by (gg_0, l_0).) This is also true in the measurable (ergodic) case.

We state the next example in the measurable case. Let M be a G-space with an ergodic G-invariant probability measure μ. Let $\rho : G \to GL(V)$ be a representation of G on the finite dimensional (real) vector space V. Denote by H the Zariski closure of $\rho(G)$ in $GL(V)$ and suppose that $\rho(G)$ is a subgroup of finite index in H. We assume moreover that H is generated by algebraic 1-parameter subgroups. Then H is the algebraic hull of the G-action by bundle automorphisms of the (trivial) principal H-bundle $P = M \times H$ given by $g(x, h) := (gx, \rho(g)h)$ (see [3]).

Before expanding on the previous example, we introduce some notations and terminology. It will be assumed here that all our bundles, maps, and actions are smooth, even if this is not an essential hypothesis. Let $p : E \to M$ be a vector bundle over a manifold M. We recall that given another manifold M' and any map $\pi : M' \to M$ one can define a vector bundle, π^*E, over M', called the *pull-back of* E under π, as follows:

$$p_1 : \pi^*E = \{(m', e) \in M' \times E \mid \pi(m') = p(e)\} \to M' : (m', e) \mapsto m' \,.$$

super-rigidity

The map $p_2 : (m', e) \mapsto e$ is a vector bundle map from $\pi^* E$ into E. If $U \subset M$ is an open subset, and i is the inclusion map, then the pull-back $i^* E$ is (isomorphic to) the restriction of E to U. It will be denoted $E|_U$.

Let now G be a Lie group that acts both on M and on M' by smooth diffeomorphisms in such a way that π is an equivariant map. There is a natural and unique way to define a G-action on the pull-back of E such that p_2 is also G-equivariant: $g(m', e) = (gm', ge)$.

Let $\rho : G \to GL(n, \mathbb{R})$ be a homomorphism and introduce the vector bundle $p : E := (G \times \mathbb{R}^n)/\Gamma \to M := G/\Gamma$, where Γ is a lattice in G. Γ acts on $G \times \mathbb{R}^n$ by

$$(g, x)\gamma := (g\gamma, \rho(\gamma)^{-1} x) .$$

If Γ' is a subgroup of finite index in Γ, then $M' = G/\Gamma'$ is a finite covering of M and the pull-back of E under the covering map is naturally isomorphic to the vector bundle $E' := (G \times \mathbb{R}^n)/\Gamma'$. The isomorphism is given by

$$(g, x)\Gamma' \mapsto (g\Gamma', (g, x)\Gamma) .$$

We will call an action of a Lie group G on a vector bundle $p : E \to M$ *topologically irreducible* if the action on M is topologically transitive and the following condition holds for any G-invariant open dense subset U: If $\pi : U' \to U$ is any finite covering and π is G-equivariant for a G-action on U' which is also topologically transitive, then the pull-back of $E|_U$ to U' does not admit a proper G-invariant subbundle. The next proposition provides an example. (Notice that the action in this case is transitive.)

Proposition 6.1.2 *Let $G = SL(3, \mathbb{R})$ (or any noncompact, connected, simple real algebraic group), Γ any lattice in G. Suppose that ρ is an algebraic (rational) homomorphism of G into $GL(n, \mathbb{R})$ that defines an irreducible representation of G on $\mathbb{R}^n$. Then the vector bundle $(G \times \mathbb{R}^n)/\Gamma$ does not admit a proper G-invariant continuous vector subbundle (or even a measurable one, for the Haar measure class on G/Γ).*

Proof. If the conclusion did not hold, the algebraic hull of the action would be contained in the stabilizer in $GL(n, \mathbb{R})$ of a proper subspace of $\mathbb{R}^n$ (regarded as a point in a real Grassmannian variety.) Denoting by L this stabilizer, then L is a real algebraic group such that (some conjugate of) L contains the Zariski closure of $\rho(\Gamma)$ (the algebraic hull of the action, as was already pointed out.) Therefore, $\rho^{-1}(L)$ is an algebraic subgroup of G that contains Γ. But by the Borel density theorem, Γ is Zariski dense in G, therefore $\rho(G) \subset L$. But this contradicts the assumption that ρ was irreducible. $\square$

7 Super-Rigidity

We are now in a position to discuss the main technical result of these notes. For a more general result, the details of the proof, as well as a number of applications that we will not talk about here, we refer the reader to [3] and [5]. topological super-rigidity

7.1 The Main Result

The main theorem of these notes gives a precise answer to the following vague question. Suppose that a noncompact simple Lie group G acts on a manifold M and that a subgroup $G_0 \subset G$ leaves invariant some geometric structure on M. Does G also leave invariant the same structure? If not, what can we say about the possible structures invariant under G? The answer to these questions that the theorem provides is expressed in terms of the algebraic hull of the actions. Loosely stated, the "part" of the algebraic hull of the G-action that is "not already in" the algebraic hull of the G_0-action is a homomorphic image of G.

For simplicity, we only give the smooth case. The statements for the measurable, topological and C^r cases are similar.

Theorem 7.1.1 (Topological Super-Rigidity) *Suppose that G is a connected semisimple Lie group with real rank at least 2 and that it acts by automorphisms on some principal H-bundle over a manifold M, such that the action is also smooth. We also make the following assumptions:*

1. *Every 1-parameter $\mathbb{R}$-split subgroup of G has a dense orbit in M (and a dense set of recurrent points; this is an extra condition only if the dimension of M is 1).*
2. *H is the smooth algebraic hull of the G-action*
3. *There exists a 1-parameter $\mathbb{R}$-split subgroup $G_0 \subset G$ whose smooth algebraic hull does not contain a nontrivial normal subgroup of H.*

Then there exists a (smooth) surjective homomorphism $\rho : G \to H$, a G-invariant open and dense subset $U \subset M$, and a smooth section σ of $P|_U$ such that for all $g \in G$ and $x \in U$

$$g\sigma(x) = \sigma(gx)\rho(x)\,.$$

In other words, $P|_U$ is a trivial bundle and the action on it can be described by a ρ-simple cocycle.

Important remark: Because of Moore's ergodicity theorem (Theorem 5.2.1) the first hypothesis is implied by: G preserves a probability measure on M that takes positive values on non-empty open sets, and the action is ergodic with respect to this measure.

It should also be pointed out that in the measurable form of the theorem, which is Zimmer's cocycle super-rigidity theorem, the assumption (3) is not required, since it can be shown to be a consequence of Proposition 3.5.1. Reference [3] shows how the measurable cocycle super-rigidity theorem follows from the above version of the topological super-rigidity theorem. Zimmer's proof of the cocycle super-rigidity theorem is found in [11], which in turn is based on Margulis's fundamental work, for which [9] is the main reference. The proof that will be sketched here comes from [5] and is shown in much greater detail in [3].

We will present in the next section a proof of the theorem in a special situation, which should also help clarify its hypothesis. This is a drastic specialization, for $G = SL(3,\mathbb{R})$, but it contains in its proof a fairly complete outline of the proof of Theorem 7.1.1.

Theorem 7.1.2 (Baby cocycle Super-Rigidity) *Let $G = SL(3,\mathbb{R})$ and suppose that G acts on a vector bundle $E \to M$ of fiber dimension n by smooth automorphisms such that the action is topologically irreducible (as defined immediately before Proposition 6.1.2) and leaves invariant a nonvanishing smooth n-form on E. The action on M is assumed to be ergodic with respect to a G-invariant probability measure μ, positive on nonempty open sets. Let g_0 be an $\mathbb{R}$-diagonalizable element of G different from the identity element, having a positive eigenvalue, and suppose that there exists a g_0-invariant, one-dimensional, smooth, orientable subbundle, E_0, of E. Then, there exists a G-invariant open dense (of measure 1) subset $U \subset M$ and a smooth trivialization of $E|_U$ by a frame of smooth sections $X_1, \ldots, X_n$ of E such that*

$$g_* X_i = \sum_j \rho(g)_{ij} X_j \tag{28}$$

where $\rho : G \to SL(n,\mathbb{R})$ is a homomorphism corresponding to an irreducible representation of G in dimension n. In other words, the action admits a ρ-simple frame.

8 The Proof

Before beginning the proof in earnest, we point to the following reductions and simplifications.

8.1 Preliminary Reductions

It will be convenient to express a frame field on a vector bundle F as a map $x \in M \mapsto \sigma(x)$, where $\sigma(x)$ is a linear isomorphism from $\mathbb{R}^k$ to the fiber F_x. The action of a $g \in G$ on M and on a F will be expressed simply by writing gx or gv.

Warning: Proposition 5.1.2 and Corollary 5.1.3 are employed a few times in the proof. Each time they are used we have to let go of the complement of an open invariant set of full measure in M (called U in both statements).

To streamline the prose we ignore this detail and pretend that everything is defined over M all along the way. It should be kept in mind, though, that the final conclusions will only hold on an open G-invariant subset of M of full measure. It will also be supposed, for the sake of making the discussion seem a little more concrete, that the vector bundle E is the tangent bundle, TM, of M (even though very little simplification will be gained by this assumption). In particular, rather than work with the frame bundle of E we will work with $F^1(M)$ and its associated bundles.

The smooth algebraic hull of G will be called H; P will be a smooth reduction of $F^1(M)$ with group H that is invariant under G. Notice that H is contained in $SL(n, \mathbb{R})$ since it has been assumed that the action of G preserves a volume form.

H might not be algebraically connected, which is something that can cause difficulties later. The remarks that follow will be helpful. Let H^0 be the algebraic connected component of H and define the principal bundle P/H^0 over M, whose fibers are isomorphic to the finite group H/H^0. (This is indeed a group since H^0 is normal in H and it is finite since algebraic groups have a finite number of connected components.) A principal bundle over M with discrete group is precisely a covering. The covering map is, in the present case, the projection $\pi : M' := P/H^0 \to M = P/H$. The G action is clearly also defined on this covering, since it is defined on P by bundle automorphisms. Moreover, using that H is the algebraic hull of the G-action on P, it can be shown that the G-action on M' remains topologically transitive. (As far as I can tell, it could fail to be ergodic.) We can now pull E back to M'. The assumption of topological irreducibility of E implies that π^*E also does not admit G-invariant proper subbundles.

Let G_0 denote the closed subgroup of G generated by g_0. This is abstractly isomorphic to $\mathbb{Z}$. The smooth algebraic hull of G_0 will be called H_0. It is the group of a principal bundle P_0 upon which G_0 acts by principal bundle automorphisms. It can be assumed that H_0 is a (real algebraic) subgroup of H and that P_0 is a reduction of P. The action of G_0 preserves a line field, hence its algebraic hull can be regarded as a subgroup of $SL(n, \mathbb{R})$ that stabilizes a line in $\mathbb{R}^n$. We denote by L the intersection of this stabilizer with H.

Lemma 8.1.1 *If N is a normal subgroup of H contained in L, then $N \cap H^0$ is a subgroup of $\{\pm I\}$.*

Proof. By pulling everything back to M', we may assume that H is algebraically connected and the goal reduces to proving that N is contained in $\{\pm I\}$. Notice that the inclusion of H in $GL(n, \mathbb{R})$ defines a linear representation of H which must be irreducible since we are assuming that the G-action is topologically irreducible. (An H-invariant linear subspace of $\mathbb{R}^n$ gives rise to a G-invariant subbundle of TM.) Let e denote a nonzero vector in $\mathbb{R}^n$ spanning a line that is stabilized by L. Since N is normal in H, N also stabilizes the line spanned by he for each $h \in H$. But H is irreducible, so that we can find a basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ such that each line $\mathbb{R}e_i$ is stabilized by N. In other words, with respect to this choice of basis, elements of N are represented by diagonal matrices.

Denote by V_{ij} the subset of $h \in H$ such that he_i does not lie in the linear span of the complement of $\{e_j\}$ in $\{e_1, \ldots, e_n\}$. This is the complement, S_j, of an algebraic subset of H. If V_{ij} is empty, then the linear span of $\{he_i : h \in H\}$ is contained in $\mathrm{span}(S_j)$, contradicting the irreducibility of the H-representation on $\mathbb{R}^n$. Therefore V_{ij} is non-empty for each i, j.

Since H is a connected algebraic group, it is also an irreducible variety. Consequently, it cannot be the union of the complements of the V_{ij}. In other words, there must exist h in the intersection of the V_{ij}. This means that every entry of the matrix that represents h in the basis $\{e_i\}$ is non-zero.

Let n be an arbitrary element of N and set $\bar{n} := h^{-1}nh \in N$. Then $h_{ij}n_{jj} = \bar{n}_{ii}h_{ij}$, for all i, j, so that n_{ii} does not depend on i. This shows that the elements of N are scalar matrices. On the other hand, these are matrices in $SL(n, \mathbb{R})$ since we are also assuming that the G-action preserves a volume form, so that $H \subset SL(n, \mathbb{R})$. Therefore the claim. Notice that if n is odd, we would have proved that N must be trivial. $\square$

The lemma says that the maximal normal subgroup of H contained in L must be discrete. Suppose for a moment that we have demonstrated the baby super-rigidity theorem on the principal bundle P/N rather than on P. This means that we would have found a smooth section σ of P/N (over a G-invariant open dense set!) and a homomorphism ρ of G into H/N such that $g\sigma(x) = \sigma(gx)\rho(g)$ for all g and x. On the other hand, using that G_0 preserves an orientable line and that the G-action is topologically irreducible, it is not difficult to show that the frame bundle is trivial over an open dense G-invariant set U where the section σ is defined. Therefore there must exist a smooth section of the frame bundle over U which respect to which the G-action cocycle is locally constant on $G \times U$. On the other hand, U is a connected set. (Notice that any connected component of U has to be G-invariant since G itself is connected.) Therefore the cocycle is constant and the conclusion of the theorem would hold on P as well. Taking this into account, there will not be any loss of generality in adding the hypothesis: Any normal subgroup of H contained in L is trivial.

With the remarks made so far, we can restate the baby rigidity theorem as follows:

Proposition 8.1.2 *Let P be a smooth H-reduction of the frame bundle of a manifold M, where $H \subset SL(n, \mathbb{R})$ is a real algebraic group. $G = SL(3, \mathbb{R})$ acts on P by bundle automorphisms leaving invariant, and ergodic with respect to, a probability measure μ on M positive on nonempty open sets. We assume that H is the algebraic hull of the action and that a closed subgroup G_0 generated by a diagonalizable $g_0 \in G$ with positive eigenvalues, different from the identity, preserves a smooth line field on M. Finally, suppose that the* *algebraic hull of the G_0-action does not contain a proper normal subgroup of H. Then the conclusion of the baby super-rigidity theorem holds.* H-pair

We will return to the proof of Proposition 8.1.2 after introducing some more definitions.

8.2 H-Pairs

We will need to following bit of abstraction. Let P be a principal H-bundle, where H is a real algebraic group, and suppose that a Lie group B acts on P by smooth automorphisms of P. An *H-pair* for the B-action on P consists of a pair (W, V) where V is a smooth real algebraic variety, W is a B-invariant subset of the space of all smooth H-equivariant functions from P into V, which we denote by $C^\infty(P, V)^H$, and the following properties hold:

1. The evaluation map $e_p : W \to V : \phi \mapsto \phi(p)$ is injective and $W_p := e_p(W)$ is a real subvariety of V.
2. Given any two points $p, p' \in P$, $\tau_{pp'} := e_p \circ e_{p'}^{-1} : W_p \to W_{p'}$ is an H-translation, by which we mean a map of the form $\tau_{pp'}(v) = hv$ for some $h \in H$ and all $v \in W_p$.
3. H acts transitively and effectively on V

The starting point of the proof of the rigidity theorem is the following observation:

Lemma 8.2.1 *Under the hypothesis of the baby rigidity theorem, the group G_0 admits an H-pair.*

Proof. Let ϕ the H-equivariant map from P into projective space $P^{n-1}(\mathbb{R})$ that describes the G_0-invariant line field. It takes values into a single H-orbit in $P^{n-1}(\mathbb{R})$, $V := H \cdot l$. The stabilizer of l is L, so that V is homogeneous, of the form H/L. Clearly, H acts transitively on V. It is also easy to show that if L does not contain a proper normal subgroup of H (an hypothesis that we are allowed to make as pointed out after the proof of Lemma 8.1.1), then the H-action on V is also effective. Let W be the one-point set consisting of ϕ. W is a G_0-invariant set since ϕ is G_0-invariant, and the evaluation maps are obviously injective. Since H acts transitively on V, it is also clear that the maps $\tau_{pp'}$ are H-translations. □

Implicit in the definition of an H-pair for B is a family of homomorphisms of B parametrized by P. In fact, for each $p \in P$ let F_p be the subgroup of H that fixes W_p pointwise, that is, the group consisting of all $h \in H$ such that $hv = v$ for each $v \in W_p$. Also define the group N_p comprising all $h \in H$ such that $hW_p = W_p$. Both groups are real algebraic groups and F_p is normal in N_p. The collection of all H-translations from W_p into V is naturally identified with the quotient H/F_p: $hv = h'v$ for all $v \in W_p$, if and only if $h \in F_p$.

Moreover, for each $p \in P$ and each $b \in B$, $\tau_{b^{-1}p,p} : W_p \to V$ is translation by some h in N_p, since W is B-invariant. Therefore, fixing $p \in P$, we associate to each $b \in B$ a unique element of N_p/F_p, which we denote by $\rho_p(b)$.

Proposition 8.2.2 *The map $\rho_p : B \to N_p/F_p$ is a smooth homomorphism.*

Proof. The details are in [3, 10.2.3]. The "multiplicativity" of ρ follows from the remark: for all $b_1, b_2 \in B$ and $\phi \in W$,

$$\rho_p(b_1b_2)e_p(\phi) = e_p((b_1b_2)\cdot\phi) = \rho_p(b_1)e_p(b_2\cdot\phi) = \rho_p(b_1)\rho_p(b_2)e_p(\phi)$$

where $b \cdot \phi := \phi \circ b^{-1}$. □

It is also interesting to observe what having an H-pair for G itself amounts to:

Lemma 8.2.3 *If under the conditions of the baby rigidity theorem G admits an H-pair, then the conclusion of the theorem holds.*

Proof. Let (W, V) be an H-pair for G. Fix a point $q \in P$ and let W_q, N_q, F_q, and $\rho_q : G \to N_q/F_q$ be as defined before. Let $\Psi_q : P \to H/F_q$ be the map that assigns for each $p \in P$ the H-translation τ_{pq}. A simple consequence of the definitions is that

$$\Psi_q(gp) = \Psi_q(p)\rho_q(g)^{-1}$$

for $p \in P$ and $g \in G$. Let $\Pi : H/F_q \to H/N_q$ be the natural projection. Since ρ_q takes values into N_q/F_q, the composition $\Pi \circ \Psi_q$ is G-invariant, so that we obtain a G-invariant N_q-reduction of P. But H is the algebraic hull of the G-action on P, therefore $N_q = H$. Furthermore, H acts transitively on V and since N_q leaves W_q invariant as a set, we have $W_q = V$; the condition that H acts effectively on V implies that F_q is the trivial group. Consequently, we obtain an H-equivariant map $\Psi_q : P \to H$ that is also G-equivariant, where G acts on H by means of the homomorphism ρ_q. But such an H-equivariant map is precisely equivalent to a section of P that transforms under G in the desired way. □

Given the previous two lemmas, the strategy for proving the baby rigidity theorem is now clear: Starting with an H-pair for G_0, we must wiggle our way into producing an H-pair for G.

In the remainder of this section we state without proof some general facts and a few more definitions concerning H-pairs. The proofs can be found in [3] or in [5].

We say that an H-pair (W_1, V) (for a group B) is contained in another H-pair (W_2, V) if $W_1 \subset W_2$. (Some definitions and statements made here would take a more complicated form if we took into account the proper domains of the maps and actions involved. We remind the reader of the warning at the start of Section 8.1.) An H-pair (W, V) for B is said to be *maximal* if it is equal to any other H-pair for B in which it is contained. maximal H-pair invariant H-pair

An H-pair (W, V) for B will be called *invariant* if $b \cdot \phi = \phi$ for all $\phi \in W$. Recall that the B-action on $W \subset C^\infty(P, V)^H$ is defined by $b \cdot \phi := \phi \circ b^{-1}$. If A is a subgroup of B, we say that the H-pair for B is $\mathbb{R}$-*split for* A if for each $p \in P$ and any given linear representation $R : N_p/F_p \to GL(m, \mathbb{R})$, we have that $R \circ \rho_p|_A$ is a rational representation and $(R \circ \rho_p)(a)$ is diagonalizable with real eigenvalues, for each $a \in A$.

Proposition 8.2.4 *Let $\mathcal{P} = (W, V)$ be an H-pair for B. Then*

1. *$\mathcal{P}$ is contained in a maximal H-pair for B;*
2. *If $\mathcal{P}$ is invariant, it is contained in a maximal invariant H-pair for B;*
3. *If $\mathcal{P}$ is $\mathbb{R}$-split for a subgroup $A \subset B$, it is contained in a maximal, $\mathbb{R}$-split for A, H-pair for B.*

Proof. This is Proposition 10.2.5 of [3], modulo the issue of restriction to invariant open dense sets that is being ignored here and is taken into account there. □

Suppose now that B is subgroup of a Lie group G that acts on P by bundle automorphisms. We say that another subgroup $Z \subset G$ centralizes B if each element of Z commutes with every element of B.

Proposition 8.2.5 *Suppose that (W, V) is a maximal invariant H pair for B and that the B-action on M is topologically transitive. Then (W, V) is also an H-pair (not necessarily maximal) for any subgroup $Z \subset G$ that centralizes B. The same holds if (W, V) is maximal $\mathbb{R}$-split for some subgroup $T \subset B$ that also acts topologically transitively on M.*

Proof. This is proposition 10.3.1 in [3]. □

Proposition 8.2.6 *Let Z be a group of H-bundle automorphisms of P commuting with a B-action. The action of B on M is assumed to be topologically transitive and we suppose that there exists an H-pair $(W, H/H_0)$ for B. Then there exists an H-pair $(W', H/F)$ for Z such that F is a subgroup of H_0. Furthermore, there is $\varphi \in W'$ such that $\pi \circ \varphi \in W$, where π is the natural projection from H/F onto H/H_0.*

Proof. This is proposition 10.3.3 of [3]. □

8.3 Back to the Proof of 8.1.2

Notice that having a smooth G_0-invariant line field on M is equivalent to having a smooth $GL(n, \mathbb{R})$-equivariant, G_0-invariant map ϕ, from $F^1(M)$

into projective space $P^{n-1}(\mathbb{R})$, taking values into an H-orbit $H \cdot l$, where l is the line whose stabilizer group is L. It has already been noticed (Lemma 8.2.1) that the pair $(\{\phi\}, H \cdot l)$ is an H-pair for G_0.

There is no loss of generality in assuming that g_0 is contained in the diagonal group $A \subset SL(3,\mathbb{R})$. Recall the notations of Section 4.4 for the various subgroups of $SL(3,\mathbb{R})$. We also introduce the groups A_i, $i = 1, 2, 3$, which are the connected subgroups of A having Lie algebras $H_i^{\perp}$, and $\check{A}_i$, connect subgroups of A having Lie algebras:

$$H_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$H_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a \end{pmatrix} : a \in \mathbb{R} \right\}$$

$$H_3 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}.$$

By Proposition 8.2.5 (since A centralizes g_0), there exists an H-pair for A. Consequently, we obtain a common H-pair for A_1, A_2, and A_3. The same proposition, applied to each A_i, yields an H-pair for their centralizer, Z_i.

We remark that any H-pair for Z_i is $\mathbb{R}$-split for $\check{A}_i$. In fact, $\check{A}_i$ consists of the diagonal elements of a subgroup of Z_i that is locally isomorphic to $SL(2,\mathbb{R})$. The remark is then a consequence of general facts about linear representations of $SL(2,\mathbb{R})$. For more details, see [3].

Therefore, by Proposition 8.2.4, we obtain a maximal H-pair (W_i, V_i) for $\check{A}_i$, which is also $\mathbb{R}$-split for $\check{A}_i$. Recall that V_i is a single H-orbit, and consequently it has the form H/L_i for some algebraic subgroup L_i of H. We may assume that i and the (W_i, V_i) have been chosen so that L_i is minimal, in the following sense: given for some j an H-pair $(W', H/L')$ for $\check{A}_j$ that is $\mathbb{R}$-split for the same $\check{A}_j$ and such that $L' \subset L_i$, then $L' = L_i$. The existence of a pair with such a minimal L is a consequence of the descending chain condition for algebraic groups.

We claim that there is a maximal H-pair for the whole of A that is also $\mathbb{R}$-split for A. First notice that by 8.2.5 $(W_i, H/L_i)$ is also an H-pair for any subgroup of A, In particular for the connected subgroup of $\check{A}_j$ for $j \neq i$. Proposition 8.2.6 now yields an H-pair $(W', H/F)$ for Z_j, with $F \subset L_i$. On the other hand, $\check{A}_j$ is contained in a subgroup of Z_j that is locally isomorphic to $SL(2,\mathbb{R})$. As already pointed out previously, $(W', H/F)$ must also be $\mathbb{R}$-split for $\check{A}_j$. Therefore $F = L_i$, due to the minimality of L_i. By 8.2.6, and the fact that $F = L_i$, we also have that $W' \cap W_i$ is not empty. Since both $(W_i, H/L_i)$ and $(W', H/L_i)$ are H-pairs for A, the intersection $(W_0, H/L_i)$ is also an H-pair for A. We claim that $(W_0, H/L_i)$ is also $\mathbb{R}$-split for A. But the Lie algebras of A_i and of A_j together span the Lie algebra of A. As A is Abelian, the pair is $\mathbb{R}$-split for A.

Denote by (W, V) a maximal $\mathbb{R}$-split H-pair for A. We now wish to show that (W, V) is an H-pair for Z_1, Z_2 and Z_3. For each i, (W, V) is an H-pair for A that is $\mathbb{R}$-split for A_i. Let (W', V') be a maximal H-pair for A that is $\mathbb{R}$-split for A_i. Then (W', V') is an H-pair for Z_i, hence it is an H-pair for A that is $\mathbb{R}$-split for $\check{A}_i$. But A_i together with $\check{A}_i$ generate the Abelian group A, so (W', V') is also $\mathbb{R}$-split for A. By maximality of (W, V) we conclude that $(W', V') = (W, V)$ hence (W, V) is also an H-pair for Z_i. But Z_1, Z_2, Z_3 generate $SL(3, \mathbb{R})$ so (W, V) is an H-pair for $SL(3, \mathbb{R})$ as well. Lemma 8.2.3 can now be used to wrap up the proof.

9 A Few Immediate Applications

We describe in this section some of the most immediate applications of the main theorem. This is not meant to be more than a collection of illustrative examples of how the theorem can be used. The reader interested in a more representative survey of applications may wish to consult the forthcoming [7].

9.1 Invariant Volumes and Connections

We use here Theorem 7.1.1 to study the following general question. If a higher-rank semisimple Lie group acts on a manifold M and a subgroup preserves a geometric structure of some kind, what can be said about geometric structures preserved by the whole group? We consider here two examples: volume forms and connections.

But first we make the following general remark. We have introduced earlier the spaces $H^1_s(G, E)$. In the same way we define the space $\hat{H}^1_s(G, E)$ where now each cohomology class $[\theta]$ is represented by a section θ of $E|_U$, for some G-invariant open dense $U \subset M$ that depends on $[\theta]$.

Proposition 9.1.1 *Let G be a connected semisimple Lie group of real rank at least 2 that acts on a C^s vector bundle E over M by automorphisms, the action being C^s. We assume that every $\mathbb{R}$-split 1-parameter subgroup of G acts topologically transitively on M with a dense set of recurrent points (or instead suppose that the action is ergodic for an invariant probability measure which is positive on nonempty open sets.) Let J be an $\mathbb{R}$-split 1-parameter subgroup of G. Then the restriction map $\hat{H}^1_s(G, E) \to \hat{H}^1_s(J, E)$ is injective.*

Proof. Let θ be an affine 1-cocycle for G, whose restriction to J is trivial, taking values into $\Gamma^s(E|_U)$, where U is a G-invariant open dense subset of M. We form as before the G-action on $(P \times_M E)|_U$ associated to θ and remark (see Proposition 3.3.1) that the algebraic hull H_J for the J-action is contained in H. Denote by H_G a representative of the algebraic hull for the

G-action on the same bundle. It may be assumed that $H_J \subset H_G$. If N is a maximal normal subgroup of H_G contained in H_J, then by Theorem 7.1.1 H_G/N is a homomorphic image of G, hence semisimple.

The proposition is now a consequence of the following algebraic remark: if L is a subgroup of $H \ltimes_\eta V$ and N is a closed normal subgroup of L such that L/N is semisimple with finitely many connected components and N fixes a point in V, then L also fixes a point in V (see [5, Lemma 7.3]). □

From the discussion of Section 3.4 we have the following corollary.

Corollary 9.1.2 *Under the same conditions of the previous proposition, if J preserves a C^s connection over some J-invariant open dense set, then G also preserves a C^s connection on some G-invariant open dense set.*

Notice that the similar corollary for volume forms has a somewhat stronger conclusion:

Corollary 9.1.3 *Under the same conditions of the previous proposition, if J preserves a volume form on M, then the same form is also preserved by G.*

It turns out that under certain relatively general conditions (of hyperbolicity, for example) the structure (such as a connection) that is obtained on an open dense subset of M can be show to extend to the whole manifold. We refer the reader to Section 6 of [5] were this point is explained in detail.

9.2 Lattice Actions

Up to this point we have not said anything about actions of lattices in higher-rank semisimple Lie groups. This may seem odd if one remembers that Margulis's rigidity and arithmeticity theorems are really theorems about lattice groups. In principle, a theorem about lattices can generally be turned into a theorem about the ambient Lie group by the following well known *suspension construction*. If Γ is a lattice in G that acts on a manifold M, we can form the quotient $N = (G \times M)/\Gamma$ that consists of the orbits of the Γ-action on $G \times M$ defined by $(g, x)\gamma := (g\gamma, \gamma^{-1}x)$. N is a fiber bundle over G/Γ with fibers homeomorphic to M and the G-action on N given by $g[g', x] = [gg', x]$, where $[g, x]$ is the element of N represented by (g, x), has the property that the first return (holonomy) map to a fiber is (up to conjugacy) the initial Γ-action on M.

The (measurable) cocycle super-rigidity theorem can be extended to lattices by this procedure (see [11]). The topological super-rigidity theorem also admits a version for lattices, but in applications a difficulty immediately arises: to prove that a hypothesis made for the lattice-action continues to hold for the suspension. It is not difficult to show that if Γ acts ergodically on M with respect to an invariant probability measure μ, then G also acts ergodically on N for the (essentially) product of the G-invariant probability measure on $G\backslash\Gamma$ and μ. On the other hand if the dynamics of Γ has some form of hyperbolicity it can be much harder to prove that the suspension has a similar property. There are, nevertheless, a number of classification theorems about smooth actions of lattices that use this idea as a starting point. Some of the best so far are proved by Margulis and Qian (see [10]).

9.3 Nonstationary Linearization

We only make a few comments on this topic. If a group G acts twice differentiably on a manifold M leaving invariant a C^1 connection on TM, and if there exists a (say measurable) trivialization σ of $F^1(M)$ for which the derivative cocycle of the G-action is ρ-simple, then it is essentially immediate to verify that $x \in M \mapsto \exp_x \circ\sigma(x)$ defines a nonstationary linearization of the action in the sense defined in Section 2.2. On the other hand, for single elements of G that admit uniform hyperbolicity both invariant connection (with low regularity) and some form of "partial ρ-simplicity" (like the Sternberg linearization) can often be obtained by methods of hyperbolic dynamics. The topological super-rigidity theorem can then be used to extend these invariant structures for subgroups to corresponding structures to the whole group. This type of argument is elaborated (and applied to classification problems of higher-rank lattice actions) in [4]. The problem of finding invariant locally homogeneous structures from this point of view and in the spirit of Section 2.2 (using Cartan connections) has not yet been developed but some closely related ideas can be found in [6].

9.4 Orbit Equivalence

This is one of the subjects to which the cocycle super-rigidity theorem was first applied by Zimmer. We refer to his book [11] for some well known theorems in this direction. We also refer the reader to the forthcoming [7], which will also contain more recent applications.

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