Bounded Representations of Amenable Groupoids and Transference

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Abstract

We extend the method of transference of Coifman and Weiss to bounded representations of amenable groupoids on Banach bundles.

1 Introduction

Let G be a locally compact, second countable group and λ a Haar measure on G. Given $k \in L^1_\lambda(G)$ and $f \in L^p_\lambda(G)$, one defines the convolution $k * f \in L^p_\lambda(G)$ by

$$(k * f)(g) = \int_G k(h) f(h^{-1}g) d\lambda(h).$$

The norm $N_p(k)$ of the operator $f \mapsto k * f$ on $L^p_\lambda(G)$ satisfies $N_p(k) \leq \|k\|_1$, where $\|k\|_1$ is the L^1-norm of k. It is possible, however, for $N_p(k)/\|k\|_1$ to be arbitrarily small, or even to have $N_p(k)$ finite for a nonintegrable k. To wit, the singular integral operators of Calderón and Zygmund, such as the Hilbert transform, have that property.

The method of "transference," as formulated by Coifman and Weiss in [2], allows one to preserve the norms $N_p(k)$ for a class of operators that arise from bounded representations of an amenable group G. More precisely, let X be a fixed Banach space and $L^p_\lambda(G, X)$ the space of X-valued L^p-functions on G. Denote by $N_{p,X}(k)$ the norm of the operator $f \mapsto k * f$ on $L^p_\lambda(G, X)$, for $k \in L^1_\lambda(G)$, and consider a strongly continuous representation $Q : G \to \mathcal{B}(X)$ of G on X, where $\mathcal{B}(X)$ denotes the space of bounded linear operators. Suppose that the operators $Q(g)$ are uniformly bounded and denote $C_Q = l.u.b.\{\|Q(g)\| : g \in G\} < \infty$. Define $H_k \in \mathcal{B}(X)$ as

$$H_k = \int_G k(g) Q(g) d\lambda(g).$$

Theorem 1.1 (Coifman-Weiss [2], [1]) *Let G be a locally compact, second countable, amenable group and X a Banach space. Let Q be a uniformly bounded representation of G on X. Let H_k and C_Q be as defined above, where $k \in L^1_\lambda(G)$. Then for $1 \leq p < \infty$, the operator H_k satisfies*

$$\|H_k\| \leq C_Q^2 N_{p,X}(k).$$

When X is a non-trivial closed subspace of a Lebesgue space $L^p_\mu(M)$, where (M, μ) is a measure space, then $N_{p,X}(k) = N_p(k)$.

The reader is referred to [2] and [1] for applications of the transference method to various branches of analysis.

The present paper extends Theorem 1.1 to representations of amenable groupoids on Banach bundles. (Note, however, the remark at the beginning of the next section concerning the definition of amenability.)

$K_I(\mathcal{G})$ will denote a (convolution) Banach $*$-algebra which is contained in $L^1_\nu(\mathcal{G})$ and coincides with $L^1_\nu(\mathcal{G})$ when $\mathcal{G}$ is a group with Haar measure ν. It is defined later in the paper, along with the other concepts and notations referred to here.

Theorem 1.2 *Let $\mathcal{G}$ be a locally compact groupoid and (ν, μ) a Haar-measure on $\mathcal{G}$. Assume that $(\mathcal{G}, \mu)$ is R-amenable. Let Q be a uniformly bounded representation of $\mathcal{G}$ on a Banach bundle X. Then for any $k \in K_I(\mathcal{G})$, $1 \leq p < \infty$, and L^p_μ-section ξ of X,*

$$\|H_k \xi\|_p \leq X_Q^2 N_{p,X}(k) \|\xi\|_p.$$

When the fibers of X are closed subspaces of $L^p_m(M)$ for some measure space (M, m), one has $N_{p,X}(k)$.

Let G be a second countable locally compact group, not necessarily amenable, and S a G-space. Recall that a *cocycle* of G over the G-action on S with values in another group L is a map $\alpha : S \times G \to L$ such that for all $s \in S$ and $g_1, g_2 \in G$,

$$\alpha(s, g_1 g_2) = \alpha(s, g_1)\alpha(s g_1, g_2).$$

Thus α is a homomorphism of the groupoid $\mathcal{G} = S \times G$ defined in Example 2.2, into the group L. Set, now, $S = H \backslash G$, where H is a closed subgroup of G and G acts on the quotient by right translations. Let λ be a Haar measure on G. It is known that $\mathcal{G}$ is amenable iff H is an amenable group (cf. [10]. R-amenability is claimed in [7, 3.10].) For example, when $G = SL(2, \mathbb{R})$

and H is the upper triangular group, $H\backslash G$ can be identified with the circle $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and G acts on $\bar{\mathbb{R}}$ by fractional linear transformations.

In the corollary, $\mathrm{Iso}(E)$ denotes the group of isometries of a Banach space E.

Corollary 1.3 *Let $S = H\backslash G$, where H is an amenable subgroup of G, μ is any probability measure on S and λ is a left Haar measure on G. Then for any cocycle $\alpha : S \times G \to Iso(E)$ taking values into the group of isometries of a Banach space E and $k \in K_I(S \times G)$, the operator U_k on $L^p_k(S, E)$ defined by*

$$U_k(s) = \int_G k(s, g)\alpha(s, g)d\lambda(g)$$

has norm bounded above by the convolution p-norm $N_{p,E}(k)$, where $1 \leq p < \infty$.

As another illustration of the theorem, consider a C^1 foliation $\mathcal{F}$ of a smooth manifold M. $\mathcal{F}$ defines an equivalence relation on M and we let $\mathcal{G} \subset M \times M$ denote the graph of that equivalence relation.

Let H denote a homomorphism of $\mathcal{G}$ into the group of diffeomorphisms $\mathrm{Diff}^1(N)$ of another smooth manifold N. Thus for every $(x, y) \in \mathcal{G}$, $H(x, y) \in \mathrm{Diff}^1(N)$ and $H(y, x) = H(x, y)^{-1}$, $H(x, z) \circ H(z, y) = H(x, y)$ for any x, y, z in the same equivalence class. For example, one may have a "foliated imbedding" of $\mathcal{F}$ into a foliated bundle with leaves of same dimension as $\mathcal{F}$ and fibers diffeomorphic to N. H then corresponds to the holonomy transformations on N.

H induces a homomorphism Q of $\mathcal{G}$ into the group of isometries of $E = L^p_{\mu'}(N)$, for some smooth measure μ' on N:

$$Q(x, y)g(z) = \delta(x, y)(z)^{\frac{1}{p}} g(H(y, x)(z)),$$

where $\delta(x, y)$ is the Radon-Nikodym derivative $\delta(x, y) = d(H(x, y)_*\mu')/d\mu'$.

Suppose that $\mathcal{F}$ with its smooth holonomy invariant measure class is amenable (cf. [11]) and let $L^p_\mu(M, E)$ denote the space of L^p-functions on M taking values in the Banach space E. By an L^p-bounded integral operator on $\mathcal{F}$ we mean a measurable family of integral operators on the leaves of $\mathcal{F}$ parametrized by M such that for a. e. $y \in M$, $T(y)$ is a (possibly singular) integral operator on the L^p-space of the leaf through y such that their norms are essentially bounded (cf. section 3). Let $m_{[x]}$ denote the Riemannian metric on M.

Corollary 1.4 *Let k be the kernel of an L^p-bounded integral operator on an amenable C^1 foliation $\mathcal{F}$ of a smooth manifold M, where $1 \leq p < \infty$. Define $T_y : L^p_{m_{[y]}}([y]) \to L^p_{m_{[y]}}([y])$ such that*

$$(T_y f)(x) = \int_{[y]} k(x,z)f(z)dm_{[y]}(z).$$

Let N be a smooth manifold and μ' a smooth measure on N. Let Q be a representation of the graph $\mathcal{G}$ of $\mathcal{F}$ by isometries of the Lebesgue space $E = L^p_{\mu'}(N)$. Then the operator U_k on $L^p_\mu(M,E)$ defined by

$$(U_k F)(x) = \int_{[x]} k(x,y)Q(x,y)F(y)dm_{[x]}(y)$$

has norm bounded above by the essential supremum of $\|T_y\|_p$, $y \in M$.

2 Definitions and Proof

The definition of amenability for groupoids that we use here is due to J. Renault, [7]. A (possibly) less restrictive definition which is generally accepted as the standard one was formulated by R. Zimmer (see, for example, [10]) and it is to the best of our knowledge an open question whether or not the two definitions are equivalent. We refer to the two notions as R-amenability and Z-amenability. We do not know whether Theorem 1.2 can be proved for general Z-amenable groupoids. We omit the prefix whenever we know that the two definitions agree for a particular class of examples.

The notation for what follows is mostly taken from [7]. Let $\mathcal{G}$ be a locally compact groupoid with unit space $\mathcal{G}^0$. The domain and range maps are denoted by $d, r : \mathcal{G} \to \mathcal{G}^0$. Define $\mathcal{G}^u = r^{-1}(u)$, $\mathcal{G}_v = d^{-1}(v)$ and $\mathcal{G}^u_v = \mathcal{G}^u \cap \mathcal{G}_v$. Let $\{\lambda^u\}_{u \in \mathcal{G}^0}$ be a left-Haar system on $\mathcal{G}$. Together with a probability measure μ on $\mathcal{G}^0$ it yields a measure ν on $\mathcal{G}$ defined as

$$\nu(A) = \int_{\mathcal{G}^0} \lambda^u(A \cap \mathcal{G}^u)d\mu(u).$$

As in [7] we require that the measure μ be *quasi-invariant* in the sense that the measure class C of ν is invariant under the inverse map on $\mathcal{G}$. We note that when μ is quasi-invariant, (ν, μ) defines a Haar measure in the sense of [4] for the measure groupoid $(\mathcal{G}, C)$. The existence of Haar measures for measure groupoids is established by Hahn in [4].)

Definition 2.1 ([7], Lemma 3.4) *$(\mathcal{G},\mu)$ is* R-amenable *if there exists a net g_δ of non-negative, compactly supported, continuous functions on $\mathcal{G}$ such that the following holds. Define for $u \in \mathcal{G}^0$ and $\eta \in \mathcal{G}$*

$$A_\delta = \int_{\mathcal{G}^u} g_\delta(\sigma) d\lambda^u(\sigma)$$
$$B_\delta(\eta) = \int_{\mathcal{G}^{r(\eta)}} |g_\delta(\eta^{-1}\sigma) - g_\delta(\sigma)| d\lambda^{r(\eta)}(\sigma).$$

Then A_δ converges to 1 in the weak-topology of $L^\infty_\mu(\mathcal{G}^0)$ and B_δ converges to 0 in the weak*-topology of $L^\infty_\nu(\mathcal{G})$.*

Example 2.2 (Cf. [7] 3.10) Suppose that a second countable, locally compact group G acts on the space S on the right and denote the image of $u \in S$ under $g \in G$ by ug. Set $\mathcal{G} = S \times G$. The units of $\mathcal{G}$ are (u,e) for $u \in S$ and e the unit element of G (so that we may identify $\mathcal{G}^0 = S$). Define the groupoid structure as follows: (u,g) and (v,h) are composable if and only if $v = ug$ and

$$(u,g)(v,h) = (u,gh),\ (u,g)^{-1} = (ug, g^{-1}),\ r(u,g) = u,\ d(u,g) = ug.$$

A Haar measure for the above groupoid can be obtained from a left Haar measure on G and an arbitrary measure on S (cf. [7] 3.21). If λ denotes a leaft Haar measure on G, one takes $\lambda^u = \delta_u \otimes \lambda$. When G is amenable, $\mathcal{G}$ is also amenable for any quasi-invariant measure (defined as in [7]) measure on S. It is possible, however, for a non-amenable group to act amenably on somem space S. In fact, if H is a closed subgroup of G, $\mathcal{G} = H\backslash G \times G$ is amenable if and only if H is an amenable group. A representation of $\mathcal{G}$ corresponds to a *cocycle* over the group S, taking values in the group of bounded invertible transformations of a Banach space. (For other examples, we refer the reader to [7].)

Definition 2.3 *$L^{1,\infty}_\nu(\mathcal{G})$ denotes the mixed Lebesgue space of measurable functions k on $\mathcal{G}$ with norm $\|k\|_{1,\infty}$ defined as the essential supremum over $(\mathcal{G}^0,\mu)$ of $u \mapsto \int_{\mathcal{G}^u} |k(\sigma)| d\lambda^u(\sigma)$.*

Definition 2.4 *We say that a surjective map $p : X \to \mathcal{G}^0$ is a* Banach bundle *with base $\mathcal{G}^0$ if for each $u \in S$, $p^{-1} = X_u$ (the fiber over u) is a Banach space (with norm $\| \|_u$) and there exists a countable measurable partition $\mathcal{A}$ of $\mathcal{G}^0$ such that for each $A \in \mathcal{A}$, $p^{-1}(A)$ is (measurably isomorphic, with isometries between fibers, to) the trivial bundle $A \times E$, for a Banach space*

E. Define the space $L^p_\nu(\mathcal{G}^0, X)$ of L^p-sections of X for a Borel measure μ on $\mathcal{G}^0$. Also define the space $L^p_\nu(\mathcal{G}, X)$ of measurable functions $f : \mathcal{G} \to X$ such that $f(\sigma) \in X_{d(\sigma)}$ and $\sigma \mapsto \|f(\sigma)\| \in L^p_\nu(\mathcal{G})$. (Given a Banach space E, viewed as a bundle over a single point 0, one may define a Banach bundle of $\mathcal{G}^0$ by pull-back under the constant map $\mathcal{G}^0 \to \{0\}$.)

Consider the groupoid $\mathcal{B}(X)$ defined as follows: its set of units is $\mathcal{B}(X)^0 = \mathcal{G}^0$ and for all $u, v \in \mathcal{G}^0$, $\mathcal{B}(X)^u_v$ consists of all bounded linear operators from X_v to X_u.

Definition 2.5 *A* (uniformly) bounded representation *of $\mathcal{G}$ on a Banach bundle $p : X \to \mathcal{G}^0$ is a groupoid homomorphism $Q : \mathcal{G} \to \mathcal{B}(X)$ whose elements are uniformly bounded. More precisely, we assume that $Q(\sigma\eta) = Q(\sigma)Q(\eta)$ for all $\sigma, \eta \in \mathcal{G}$ such that $d(\sigma) = r(\eta)$, $Q(\sigma^{-1}) = Q(\sigma)^{-1}$ and*

$$l.u.b.\{\|Q(\sigma)\| : \sigma \in \mathcal{G}\} = C_Q < \infty.$$

Definition 2.6 *The convolution of functions on $\mathcal{G}$ is defined by*

$$(k * f)(\eta) = \int_{\mathcal{G}^{r(\eta)}} k(\sigma) f(\sigma^{-1}\eta) d\lambda^{r(\eta)}(\sigma).$$

Denote by $N_{p,X}(k)$ the norm of the operator $F \mapsto k * F$ on $L^p_\nu(\mathcal{G}, X)$. Also define on $L^p_\mu(\mathcal{G}^0, X)$ the operator

$$H_k \xi(u) = \int_{\mathcal{G}^u} k(\sigma) Q(\sigma) \xi(d(\sigma)) d\lambda^u(\sigma).$$

Fix a Haar measure (ν, μ) on $\mathcal{G}$. Denote by $\Delta = (d\nu^{-1}/d\nu)^{-1}$ its modular function, which is a groupoid homomorphism from $\mathcal{G}$ into $\mathbb{R}^*_+$. Define the conjugation $f^*(\sigma) = \overline{f(\sigma^{-1})}\Delta(\sigma)^{-1}$, of functions on $\mathcal{G}$. It is immediate that $\|f^*\|_1 = \|f\|_1$. Define

$$\|f\|_I = \max\{\|f\|_{1,\infty}, \|f^*\|_{1,\infty},\}.$$

For $1/p + 1/q = 1$ and $f \in L^1_\nu(\mathcal{G})$, define

$$\|f\|^*_{p,q} = \sup\left\{\int_{\mathcal{G}} |f(\sigma)g(d(\sigma))h(r(\sigma))| d\nu(\sigma) : \int_{\mathcal{G}^0} |g|^p d\mu = \int_{\mathcal{G}^0} |h|^p d\mu = 1\right\}.$$

The definitions immediately imply $\|f^*\|^*_{p,q} = \|f\|^*_{p,q}$ and $\|f^*\|_I = \|f\|_I$.

The next proposition collects some basic facts about these various norms.

Proposition 2.7 *The norms defined above satisfy* $\|f\|_1 \le \|f\|^*_{p,q} \le \|f\|_I$. *Set* $K^*_{p,q}(\mathcal{G}) = \{k \in L^1_\nu(\mathcal{G}) : \|k\|^*_{p,q} < \infty\}$, $K_I(\mathcal{G}) = \{k \in L^1_\nu(\mathcal{G}) : \|k\|_I < \infty\}$. *Then* $(K^*_{p,q}(\mathcal{G}), \| \ \|^*_{p,q})$ *and* $(K_I(\mathcal{G}), \| \ \|_I)$ *are complete normed spaces.* $K_I(\mathcal{G})$ *is closed under convolution, hence defines a Banach* $*$*-algebra. When* $\mathcal{G}$ *is a group (thus* $\mathcal{G}^0 = \{e\}$*),* $K_I(\mathcal{G}) = K^*_{p,q}(\mathcal{G} = L^1_\nu(\mathcal{G})$.

Proof. All the claims can be found in [5] for $p = q = 2$. The general case requires only obvious modifications of the proofs given there. □

Proposition 2.8 *Let* $1 \le p < \infty$, $1/p + 1/q = 1$ *and* X *a Banach bundle. Then for any* $k \in K^*_{p,q}(\mathcal{G}$, $F \mapsto k*F$ *defines a bounded operator on* $L^p_\nu(\mathcal{G}, X)$ *with norm* $N_{p,X}(k) \le \|k\|^*_{p,q}$.

Proof. It suffices to show that for any $F \in L^p_\nu(\mathcal{G}, X)$ and $G \in L^q_\nu(\mathcal{G}, X^*)$

$$\int_{\mathcal{G}} |\langle G, k * F\rangle| d\nu \le \|k\|^*_{p,q} \|F\|_p \|G\|_q.$$

This, in turn, is a consequence of the next inequality, where $f = \|F\|$ and $g = \|G\|$:

$$\int_{\mathcal{G}} |(|k| * f)g| d\nu \le \|k\|^*_{p,q} \|f\|_p \|g\|_q.$$

The proof now follows the steps of the proof of Lemma 2.1 of [5], with the obvious modification that Cauchy-Schwartz inequality is replaced with Hölder's inequality. □

Proposition 2.9 *Suppose that the fibers of the Banach bundle* X *are closed subspaces of a Lebesgue space* $L^p_m(M)$. *Let* $k \in K_I$ *and* $F \in L^p_\nu(\mathcal{G}, X)$. *Then* $N_p(k) = N_{p,X}(k)$.

Proof. Let F be a function of type $\sum_{l=1}^\infty f_l \otimes \chi_{A_l}$, jointly integrable on $\mathcal{G} \times M$, where each f_l is an element of $L^p_m(M)$. Denote $F_x(\sigma) = F(\sigma)(x)$, for $(\sigma, x) \in \mathcal{G} \times M$. Then,

$$\begin{aligned}
\|k * F\|^p &= \int_{\mathcal{G}} \int_M |(k * F)_x(\eta)|^p dm(x) d\nu(\eta) \\
&= \int_{\mathcal{G}} \int_M |(k * F_*)(\eta)|^p dm(x) d\nu(\eta) \\
&= \int_M \int_{\mathcal{G}} |(k * F_*)(\eta)|^p dm(x) d\nu(\eta) \\
&\leq \int_M N_p(k)^p \int_{\mathcal{G}} |F_x(\eta)|^p d\nu(\eta) dm(x) \\
&= N_p(k)^p \int_{\mathcal{G}} \int_M |F_x(\eta)|^p d\nu(\eta) dm(x) \\
&= N_p(k)^p \|F\|^p.
\end{aligned}$$

For a general F, the same follows by a standard approximation argument. This shows $N_{p,X}(k) \leq N_p(k)$. For the opposite inequality, let s be a section of X such that $\|s(u)\| = 1$ for all $u \in \mathcal{G}^0$. Given $f \in L^p_\nu(\mathcal{G})$, form $f \otimes s$, so that $(f \otimes s)(\sigma) = f(\sigma) s(d(\sigma))$. Then $k * (f \otimes s) = (k * f) \otimes s$, so one obtains the opoposite inequality. □

Proof of Theorem 1.2. Denote $Q_\sigma = Q(\sigma)$. Let g_δ be as in the definition of amenability. If $\eta \in \mathcal{G}^u$, one has

$$H_k \xi(u) g_\delta(\eta)^{\frac{1}{p}} = a_\delta(\eta) + b_\delta,$$

where

$$\begin{aligned}
a_\delta(\eta) &= \int_{\mathcal{G}^u} k(\sigma) Q_\sigma \xi(d(\sigma)) g_\delta(\sigma^{-1}\eta)^{\frac{1}{p}} d\lambda^u(\sigma) \\
b_\delta(\eta) &= \int_{\mathcal{G}^u} k(\sigma) Q_\sigma \xi(d(\sigma)) (g_\delta(\eta)^{\frac{1}{p}} - g_\delta(\sigma^{-1}\eta)^{\frac{1}{p}}) d\lambda^u(\sigma).
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\int_{\mathcal{G}^0} |H_k \xi(u)|^p A_\delta d\mu(u) \right)^{\frac{1}{p}} &= \left(\int_{\mathcal{G}^0} \int_{\mathcal{G}^u} |H_k \xi(u)|^p g_\delta d\mu(u) \right)^{\frac{1}{p}} \\
&\leq \|a_\delta\|_p + \|b_\delta\|_p,
\end{aligned}$$

where the left-hand side converges to $\|H_k \xi\|_p$.

Define the function

$$F(\eta) = g_\delta(\eta)^{\frac{1}{p}} Q_{\eta^{-1}} \xi(r(\eta)).$$

Then, as $Q_\sigma = Q_\eta Q_{\eta^{-1}\sigma}$ and the norm of Q_η is bounded by C, one has

$$\int_{\mathcal{G}} |a_\delta(\eta)|^p d\nu(\eta) \leq C^p \int_{\mathcal{G}} |(k * F)(\eta)|^p d\nu(\eta).$$

The function F belongs to $L^p_\nu(\mathcal{G}, X)$. It then follows that

$$\begin{aligned}\|a_\delta\|_p^p &\leq C^p (N_{p,X}(k))^p \|F\|_p^p \\ &= C^p (N_{p,X}(k))^p \int_{\mathcal{G}} g_\delta(\eta) |Q_{\eta^{-1}} \xi(r(\eta))|^p d\nu(\eta) \\ &\leq C^{2p} (N_{p,X}(k))^p \int_{\mathcal{G}} g_\delta(\eta) |\xi(r(\eta))|^p d\nu(\eta).\end{aligned}$$

Therefore, in order to prove the claim, it suffices to show that $\|b_\delta\|_p^p$ converges to 0 as $\delta \to \infty$. It is also sufficient to suppose that $\xi \in L^\infty_\mu(\mathcal{G}^0, X)$ as this space is dense in $L^p_\mu(\mathcal{G}^0, X)$. Minkowski's inequality implies that $\|b_\delta\|_p^p$ is bounded above by $\int_{\mathcal{G}^0} A(u)^p d\mu(u)$, where $A(u)$ is given by

$$\int_{\mathcal{G}^u} \left(\int_{\mathcal{G}^u} |k(\sigma) Q_\sigma \xi(d(\sigma)) (g_\delta(\eta)^{\frac{1}{p}} - g_\delta(\sigma^{-1}\eta)^{\frac{1}{p}})|^p d\lambda^u(\eta) \right)^{\frac{1}{p}} d\lambda^u(\sigma).$$

Denote $B_\delta(\sigma) = \int_{\mathcal{G}^u} |g_\delta(\eta) - g_\delta(\sigma^{-1}\eta)| d\lambda(\eta)$, and recall that B_δ converges to zero in the weak*-topology of $L^\infty_\nu(\mathcal{G}$ as $\delta \to \infty$. Using

$$|g_\delta(\eta)^{\frac{1}{p}} - g_\delta(\sigma^{-1}\eta)^{\frac{1}{p}}|^p \leq |g_\delta(\eta) - g_\delta(\sigma^{-1}\eta)|,$$

one obtains

$$\|b_\delta\|_p^p \leq C^p \|\xi\|_\infty^p \int_{\mathcal{G}^0} \left(\int_{\mathcal{G}^u} |k(\sigma)| (B_\delta(\sigma))^{\frac{1}{p}} d\lambda^u(\sigma) \right)^p d\mu(p).$$

An application of Hölder's inequality with $1/p + 1/q = 1$ now gives:

$$\begin{aligned}\|b_\delta\|_p^p &\leq C^p \|\xi\|_\infty^p \int_{\mathcal{G}^0} \left(\int_{\mathcal{G}^u} |k(\sigma)|^{\frac{1}{q}} (k(\sigma) B_\delta(\sigma))^{\frac{1}{p}} d\lambda^u(\sigma) \right)^p d\mu(u) \\ &\leq C^p \|\xi\|_\infty^p \int_{\mathcal{G}^0} \left(\int_{\mathcal{G}^u} |k(\sigma)| d\lambda^u(\sigma) \right)^{\frac{p}{q}} \left(\int_{\mathcal{G}^u} |k(\sigma)| B_\delta(\sigma) d\lambda^u(\sigma) \right) d\mu(u) \\ &\leq C^p \|\xi\|_\infty^p (\|k\|_{1,\infty})^{\frac{p}{q}} \int_{\mathcal{G}} |k(\sigma)| B_\delta(\sigma) d\nu(\sigma).\end{aligned}$$

Therefore, as B_δ approaches 0 in the weak*-topology of $L^\infty_\nu(\mathcal{G})$, as k is in $L^1_\nu(\mathcal{G})$, $\|b_\delta\|_p^p$ approaches zero.

3 Amenable Foliations

We give now one illustration of the main theorem, where the groupoid corresponds to (the graph of) an equivalence relation on a smooth manifold associated to a foliation. Let M be a C^∞ manifold and let $\mathcal{F}$ denote a C^1 foliation on M. Thus M admits a cover by open sets $\mathcal{U}$ homeomorphic to rectangles $U \times T \subset \mathbb{R}^p \times \mathbb{R}^q$ and the changes of coordinates are C^1 functions of the form $u' = \phi(u,t), t' = \psi(t)$, where ψ is a local diffeomorphism of $\mathbb{R}^q$. The plaques of $\mathcal{U}$ are given by the equation $\pi(x) = \text{const.}$, where π is the projection onto the second factor. One defines on M the topology of leaves, which has the plaques of the distinguished open sets as a basis, and calls $\mathcal{F}$ the resulting manifold. The leaves of $\mathcal{F}$ are the connected components of $\mathcal{F}$. Let now $\mathcal{G}$ denote the (graph of the) equivalence relation that $\mathcal{F}$ defines on M. Thus $\mathcal{G}$ is the subset of $M \times M$ consisting of pairs (x, y) such that x and y belong to a same leaf.

Using a partition of unit, one can construct a family of measures m_x, $x \in M$, such that (i) the support of m_x is the leaf through x, denoted $[x]$, (ii) for every continuous function of M with compact support f, the function $x \mapsto f dm_x$ is continuous and (iii) $m_x = m_y$ if x and y are in the same leaf, so that we may write $m_x = m_{[x]}$. ([8, p. 344].) (For example, one takes the Riemannian measures obtained by restricting to leaves a Riemannian metric on M. Let λ^x be the measure obtained from m_x through the covering $d : \mathcal{G}^x \to [x]$. Then $\{\lambda^*\}$ is a Haar system for $\mathcal{G}$. Given a measure μ on M one obtains a measure ν on $\mathcal{G}$ by integrating λ^x.

When $\mathcal{F}$ is C^1, one may assume, by choosing μ smooth, that the measure class of ν is invariant under the flip map $i : (x,y) \mapsto (y,x)$ on $\mathcal{G}$. In particular, ν and μ define a holonomy invariant transverse measure class.

The foliation $\mathcal{F}$ with a holonomy invariant transverse measure class is *amenable* if the equivalence relation defined by $\mathcal{F}$ on any measurable transversal of $\mathcal{F}$ is Z-amenable (cf. [10]. See also [3] and [6, p. 179].)

Proposition 3.1 *A C^1 foliation $\mathcal{F}$, with its smooth transverse measure class, is amenable iff $(\mathcal{G}, \nu, \mu)$ is R-amenable. A sufficient (but not necessary) condition for amenability is that all leaves of $\mathcal{F}$ have polynomial grouth (for the Riemannian metric on leaves induced by a Riemannian metric on M.)*

Proof. One way to see this is by using the main result of [3] as follows. Choose a measurable partition $\mathcal{P}$ of M by plaques of $\mathcal{F}$ satisfying: (i) each leaf of $\mathcal{F}$ is a (at most) countable union of elements of $\mathcal{P}$, (ii) for each

$P \in \mathcal{P}$ contained in a leaf $[x]$, $0 < m_{[x]}(P) < \infty$, and (iii) there exists a measurable transversal of $\mathcal{F}$ which intersects each $P \in \mathcal{P}$ at exactly one point. (Thus we are allowed to identify $\mathcal{P}$ with the transversal.) Define a projection $\pi : M \to \mathcal{P}$ which associates to each $x \in M$ the plaque of $\mathcal{P}$ which contains it. According to [3], the relation $P_1 \sim P_2$ iff P_1 and P_2 belong to the same leaf, is generated by a measurable nonsingular transformation T on $\mathcal{P}$: $P_1 \sim P_2$ iff there exists $n \in \mathbb{Z}$ such that $T^n(P_1) = P_2$.

We can now define the net of averaging functions as follows: for each $N \in \mathbb{N}$, set

$$g_N(x,y) = \frac{1}{2N+1} \sum_{i=-N}^{N} \frac{\chi_{T^i(\pi(x))}(y)}{m_{[x]}(T^i(\pi(x)))}$$

where χ_A indicates the characteristic function of a set A. Also define a measurable function on the graph of the equivalence relation on $\mathcal{P}$ such that $T^{n(P_1,P_2)}(P_2) = P_1$. It is now an easy calculation to show that the properties of Definition 2.1 are satisfied. In fact, A_N is identically one and $B_N(x,z)$ is bounded above by a function of the type: const.$n'(\pi(x),\pi(z))/N$, where $n'(\pi(x),\pi(z)) = N$ if $n \geq 2N$ and $n' = n$ otherwise. (g_N could now be approximated by continuous functions with compact support, although that is not necessary for the application of Theorem 1.2.)

The claim concerning polynomial growth follows from [3]. □

Definition 3.2 *Let T be an operator on $L^p_\nu(\mathcal{G}$ given as the limit of integral operators of type*

$$(T_\epsilon f)(x,y) = \int_{[y]} k_\epsilon(x,z) f(z,y) dm_{[y]}(z),$$

where the kernel k_ϵ belongs to $K_I(\mathcal{G})$, and the inequality

$$\int_{[y]} |(T_\epsilon f)(x,y)|^p dm_{[y]}(x) \leq C(y)^p \int_{[y]} |f(x,y)|^p dm_{[y]}(x)$$

holds, where $C(y)$ is an essentially bounded function on M, independent of ϵ. We refer to T as an L^p-bounded integral operator on $\mathcal{F}$.

The above T could correspond, for example, to a family T_y, $y \in M$, of singular integral operators (cf. [9]) on the leaves $[y]$ of $\mathcal{F}$, such that their norms constitute an essentially bounded function over M.

Proposition 3.3 *Let T_k be a L^p-bounded integral operator on $\mathcal{F}$ with kernel k_ϵ. Then $T_k f = k * f$ for any $f \in L^p_\nu(\mathcal{G})$, and $N_p(k)$ is bounded by the essential supremum of the function $C(y)$ given in the above definition.*

Proof. Let $\lambda_x = i_* \lambda^x$ and $\bar{\mu}$ another measure on M (in the same class as μ) such that $\nu = \int_M \lambda_x d\bar{\mu}(x)$. Also notice that $r_* \lambda_x = d_* \lambda^x = m_{[x]}$. Then, writing $\eta = (x, y)$, $\sigma = (y, z)$,

$$\begin{aligned}
\|k_\epsilon * f\|_p^p &= \int_M \int_{\mathcal{G}_y} |(k * f)(\eta)|^p d\lambda_y(\eta) d\bar{\mu}(y) \\
&= \int_M \int_{\mathcal{G}_y} |\int_{\mathcal{G}^{d(\eta)}} k(\eta\sigma) f(\sigma^{-1}) d\lambda^{d(\eta)}(\sigma)|^p d\lambda_y(\eta) d\bar{\mu}(y) \\
&= \int_M \int_{[y]} |\int_{[y]} k(x, z) f(z, y) dm_{[y]}(z)|^p dm_{[y]}(x) d\bar{\mu}(y) \\
&\leq \|C\|_\infty^p \|f\|_p^p.
\end{aligned}$$

Corollary 1.4 now follows. □

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