Fluid Flows and Fluid Circuits

Notes for Math 308 - Mathematics for the Physical Sciences

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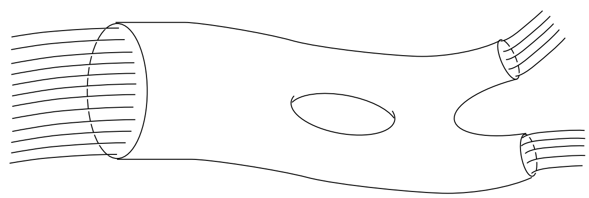
2/18/03

1 General Assumptions

The picture represents a "pipe" bounded by a surface S, through which runs some kind of fluid. The fluid flow is characterized by a vector field $\mathbf{V}$. The value $\mathbf{V}(\mathbf{x})$ at a point $\mathbf{x}$ represents the velocity of the flow at position $\mathbf{x}$.



We are going to make the following assumptions about the vector field $\mathbf{V}$:

1. The vector field is tangent to the boundary S of the pipe. Thus, if $\mathbf{n}$ denotes the unit normal vector field of S, then for each $\mathbf{x}$ on S
$$\mathbf{V}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0.$$
2. The fluid is *incompressible*. Mathematically this means that
$$\nabla \cdot \mathbf{V} = 0.$$
3. $\mathbf{V}$ is a *gradient* field; that is, for some scalar function ϕ we have
$$\mathbf{V} = \nabla \phi.$$

2 Current

We define the *current* of the fluid flow through a cross section of the pipe by

$$I(\mathcal{A}) = \iint_{\mathcal{A}} \mathbf{V} \cdot \mathbf{n} \, d\sigma.$$

Here, $\mathcal{A}$ is a cross-section surface, oriented by a unit normal vector field $\mathbf{n}$. Notice the following general fact. Let S_1 and S_2 be two cross-section surfaces. Suppose that S_3 is the part of the boundary surface of the pipe that together with S_1 and S_2 form a closed surface L. (See the next picture.) Then

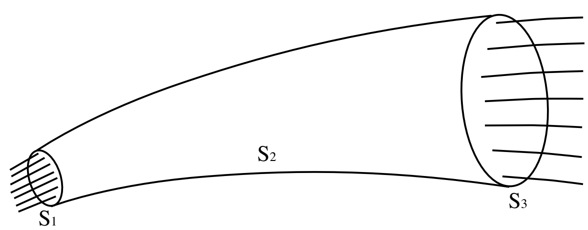
$$L = -S_1 + S_2 + S_3,$$

where the signs indicate orientation relative to the outward normal vector field of L. Then, as $\nabla \cdot \mathbf{V} = 0$, and $\iint_{S_3} \mathbf{V} \cdot \, d\sigma = 0$ (since $\mathbf{V}$ is tangent to S_3) we obtain:

$$I(S_2) = I(S_1).$$

This is because

$$I(S_2) - I(S_1) = \iint_{-S_1+S_2+S_3} \mathbf{V} \cdot \mathbf{n} \, d\sigma = \iiint \nabla \cdot \mathbf{V} \, d\tau = 0.$$



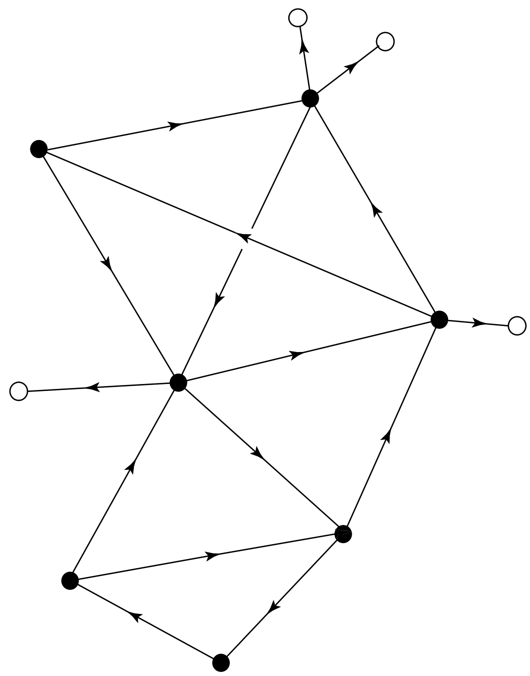
Notice that if $\mathbf{V}$ is approximately constant on cross-section surfaces and point towards the normal direction, then (since $I(A_1) = I(A_2)$)

$$V_1 A_1 = V_2 A_2,$$

where A_1, A_2 are the cross-section areas and V_1, V_2 are the magnitudes of the fluid velocity at S_1 and S_2, respectively. In words, the fluid flows more rapidly where the pipe is narrower.

3 The Circuit Approximation

Suppose that we have a system of long and narrow pipes, which we represent by a network, such as the one described by the next picture.



The network consists of the following elements:

1. A collection of *nodes*. These are the black dots representing the points of juncture of converging pipes. We will denote them by n_1, n_2, etc.

2. A collection of *open nodes*. These are the empty circles on the network. Fluid will flow in or out at these nodes. (They will be denoted by n_i, just like a regular node.)

3. A collection of pipe *branches*. These are the oriented edges of the network. The orientations are chosen in a *completly arbitrary* way. Branches will be denoted by b_1, b_2, etc. Having oriented the branches, we can speak of its initial node and end node. If a branch b has initial node n_1 and end node n_2, we also write $b = \overrightarrow{n_1 n_2}$.

To each branch, b, and each empty node, n, we associate the following data:

1. the length, $l(b)$;

2. the cross-section area, $A(b)$;

3. the current, $I(n)$, pumped (in or out) at n.

We will follow the convention that if n is the initial (resp., end) node of a terminal branch and $I(n)$ is positive, then fluid is being pumped in (resp., out). If $I(n)$ is negative, then fluid is being pumped out (resp., in).

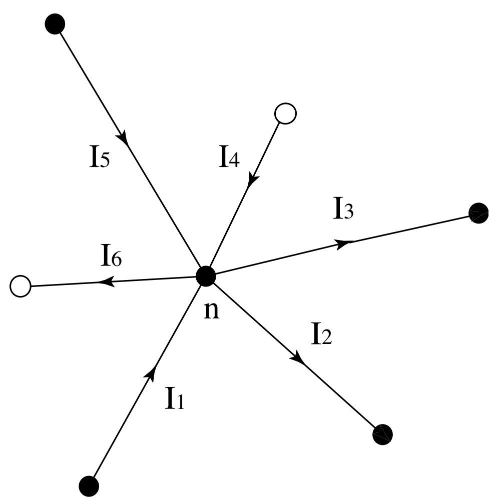
4 Kirchhoff's Current Law

Let n be a non-terminal (i.e., black) node. Suppose that $b_1, b_2, \ldots, b_k$ are all the branches that either begin or end at n. Let

$$\operatorname{sign}_n(b_i) = \begin{cases} +1 & \text{if } n \text{ is the end node of } b_i; \\ -1 & \text{if } n \text{ is the initial node of } b_i. \end{cases}$$

Then Kirchhoff's current law states that

$$\operatorname{sign}_n(b_1)I(b_1) + \operatorname{sign}_n(b_2)I(b_2) + \cdots + \operatorname{sign}_n(b_k)I(b_k) = 0.$$



For the node n in the previous picture, the equation reads:

$$I_1 - I_2 - I_3 + I_4 + I_5 - I_6 = 0.$$

This is easy to obtain as a consequence of the divergence theorem. In fact, we can apply the same argument used before to show that $I(S_1) = I(S_2)$, but now for the joining of k pipes having cross-sectional surfaces $S_1, \ldots, S_k$. I leave the details of the argument to you.

5 Ohm's Law

We have assumed that the velocity field $\mathbf{V}$ is a gradient, $\mathbf{V} = \nabla\phi$. Therefore, integrating $\mathbf{V}$ along a branch pipe b from n_1 to n_2 gives:

$$\int_b \mathbf{V} \cdot d\mathbf{r} = \phi(n_2) - \phi(n_1).$$

From the definition of current, the component of $\mathbf{V}$ along the orientation of b is $I(b)/A(b)$. It follows from the above line integral (and the approximation that $\mathbf{V}$ is constant along a branch) that

$$\frac{I(b)l(b)}{A(b)} = \phi(n_2) - \phi(n_1).$$

We define the *resistance* of a branch b by:

$$r(b) = \frac{l(b)}{A(b)}.$$

(The longer and narrower the pipe the greater the resistance.) This gives Ohm's law:

$$\Delta\phi = rI.$$

6 Kirchhoff's Potential Law

We say a closed loop, α, of the circuit is a *mesh* if it does not intersect itself except at the beginning and end nodes. For each mesh we obtain an equation involving currents. This is done as follows.

Suppose that α can be traversed by following (in the given order) the nodes

$$n_1, n_2, \ldots, n_l, n_{l+1} = n_1.$$

For each index $i = 1, 2, \ldots, l$, either $\overrightarrow{n_i n_{i+1}}$ is a branch of the circuit, or $\overrightarrow{n_{i+1} n_i}$ is. Denote the branch by b. In the first case we say that b is oriented *along* the mesh. Otherwise we say that b is oriented *against* the mesh. Now define

$$\mathrm{sign}_\alpha(b) = \begin{cases} +1 & \text{if } b \text{ is oriented along the mesh} \\ -1 & \text{if } b \text{ is oriented against the mesh.} \end{cases}$$

For each mesh α made up of branches $b_1, \ldots, b_m$, we write the equation:

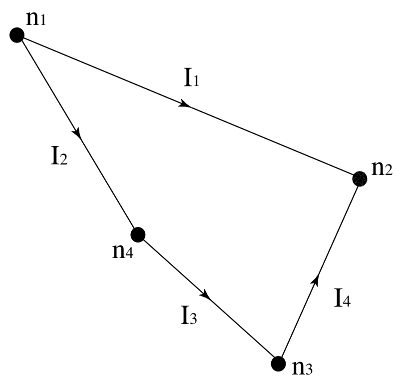
$$\mathrm{sign}_\alpha(b_1) r(b_1) I(b_1) + \cdots + \mathrm{sign}_\alpha(b_m) r(b_m) I(b_m) = 0.$$

This equation simply expresses the fact that, as $\mathbf{V}$ is a gradient field, we have:

$$\int_\alpha \mathbf{V} \cdot d\mathbf{r} = 0.$$

(I leave the details for you to check.) For example, the mesh given in the following picture (which is possibly only part of a bigger circuit) corresponds to the equation:

$$r_1 I_1 - r_4 I_4 - r_3 I_3 - r_2 I_2 = 0.$$



7 The Circuit Problem

We can now state the general problem we are interested to solve as follows. Suppose that a fluid circuit is given and that the following information is known:

1. The resistances at each branch;

2. The current (in or out) of each terminal node.

Then we wish to find the currents (hence the fluid velocities) at all branches.

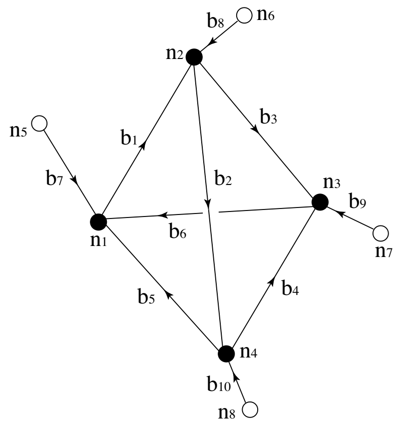
It turns out that a necessary and sufficient condition for the problem to have a solution (which will then be unique) is that the sum of the currents entering or leaving the circuit is 0. This is clearly necessary, since the fluid is incompressible. (Again, by the divergence theorem.)

Rather than prove this, we will simply work out the example given next.

8 An Example

Consider the circuit shown in the next picture. The data are the following:

$$\begin{aligned}
&r_1 = r_3 = r_5 = 1, \\
&r_2 = r_4 = r_6 = 2, r_7 = r_8 = r_9 = r_{10} = 0, \\
&I_7 = 1, I_8 = 2, I_9 = 3, I_{10} = -6.
\end{aligned}$$



The problem is to find $I_1, \ldots, I_6$.

We have one equation for each non-terminal node:

$$\begin{aligned}
n_1 &: \ I_1 - I_6 - I_5 = I_7 \\
n_2 &: \ I_2 - I_1 + I_3 = I_8 \\
n_3 &: \ I_6 - I_3 - I_4 = I_9 \\
n_4 &: \ I_4 - I_2 + I_5 = I_{10}.
\end{aligned}$$

Before writing the equations for the meshes, observe that adding the above four equations gives 0. This means that these equations are *linearly dependent.* In particular, the fourth equation is the negative of the sum of the first three. Since it adds no new information, it can be discarded.

Now, the meshes: A little thought shows that all meshes can be generated from only 3:

$$\begin{aligned}
\text{mesh}_1 &: \ b_1 + b_3 + b_6 \\
\text{mesh}_2 &: \ b_1 + b_2 + b_5 \\
\text{mesh}_3 &: \ b_5 - b_6 - b_4.
\end{aligned}$$

For example, the mesh that corresponds to $b_2 + b_4 - b_3$ can be obtained by "adding"

$$\text{mesh}_2 - \text{mesh}_1 - \text{mesh}_3.$$

(What this really means is that the corresponding current equation is linearly dependent, and can be obtained from the equations associated to the first three meshes by the sum just described.)

The corresponding current equations are:

$$\begin{aligned}
\text{mesh}_1 &: \ r_1 I_1 + r_3 I_3 + r_6 I_6 = 0 \\
\text{mesh}_2 &: \ r_1 I_1 + r_2 I_2 + r_5 I_5 = 0 \\
\text{mesh}_3 &: \ r_5 I_5 - r_6 I_6 - r_4 I_4 = 0.
\end{aligned}$$

We thus have 6 equations and 6 unknowns:

$$\begin{aligned}
\text{equ}_1 &: \ I_1 - I_6 - I_5 = 1 \\
\text{equ}_2 &: \ I_2 - I_1 + I_3 = 2 \\
\text{equ}_3 &: \ I_6 - I_3 - I_4 = 3 \\
\text{equ}_4 &: \ I_1 + I_3 + 2I_6 = 0 \\
\text{equ}_5 &: \ I_1 + 2I_2 + I_5 = 0 \\
\text{equ}_6 &: \ I_5 - 2I_6 - 2I_4 = 0.
\end{aligned}$$

I'll leave it as an exercise for you to find the values of $I_1, \ldots, I_6$.

9 The Dirichlet Problem

To be continued ...