The Foliated Liouville Problem

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General setting – harmonic version

- M – a compact connected manifold without boundary;
- $\mathcal{F}$ – a continuous foliation of M by (smooth) Riemannian manifolds.

Definition: The foliated space $(M, \mathcal{F})$ has the *Liouville property* if continuous leafwise harmonic functions on M are leafwise constant. If this property holds, we also say that $(M, \mathcal{F})$ is *harmonically simple*.

Problem: Characterize the $(M, \mathcal{F})$ that have this property.

Holomorphic version

- M – a compact connected manifold without boundary;
- $\mathcal{F}$ – a continuous foliation of M by complex manifolds.

Definition: The foliated space $(M, \mathcal{F})$ is *holomorphically simple* if continuous leafwise holomorphic functions on M are leafwise constant.

Uninteresting examples:

- If the leaves of $\mathcal{F}$, individually, do not admit bounded harmonic functions, then $(M, \mathcal{F})$ is harmonically simple. This is the case, for example, if the leaves of $\mathcal{F}$ have non-negative Ricci curvature. (S.-T. Yau: If L is a complete Riemannian manifold with non-negative Ricci curvature, then there are no non-constant bounded harmonic function.)

- In the holomorphic case, if each leaf of $\mathcal{F}$ admits $\mathbb{C}^m$ as a (holomorphic) covering space, then $(M, \mathcal{F})$ is holomorphically simple.

- If the leaves of $\mathcal{F}$ are holomorphically parallelizable (e.g., $\mathcal{F}$ is the orbit foliation of a locally free action of a complex Lie group on M), then $(M, \mathcal{F})$ is holomorphically simple.

Some "negative" results (conditions that imply the Liouville property)

Theorem 1. (Holomorphic) *If the closure of each leaf of* $(M, \mathcal{F})$ *contains (at most) countably many minimal sets. Then the foliation is holomorphically simple.*

Theorem 2. (Holomorphic) *If* $(M, \mathcal{F})$ *has codimension-one, then it is holomorphically simple.*

Theorem 3. [Garnett] (Harmonic) *If the union of the supports of harmonic measures is all of* M*, then* $(M, \mathcal{F})$ *is harmonically simple.*

Foliated bundle over S with fiber X

Let S be a compact connected Riemannian (complex, in the holomorphic case) manifold, $\tilde{S}$ its universal covering space, and γ the fundamental group of S represented as the group of deck transformations of $\tilde{S}$. Let X be a compact connected space on which Γ acts by homeomorphisms. Let $M = (\tilde{S} \times X)/\Gamma$ denote the space of orbits for the action of Γ on $\tilde{S} \times X$ defined by $(s, x) \cdot \gamma := (s\gamma, \gamma^{-1}(x))$. Then M is foliated by Riemannian (complex) manifolds locally isomorphic to S.

If ρ denotes the action of Γ on X, the associated foliated bundle will be written $(M_\rho, \mathcal{F}_\rho)$.

Of particular interest: S is a Riemann surface of genus $g \geq 2$.

"Negative" results for foliated bundles

Let S be a compact Riemann surface of genus at least 2. If G is an algebraic group, let $Hom(\Gamma, G)$ denote the variety of homomorphisms from Γ into G.

Theorem 4. *Let $(M_\rho, \mathcal{F}_\rho)$ be the foliated bundle over S with fiber $P^{n-1}(\mathbb{C})$ and action induced by $\rho : \Gamma \to GL(n, \mathbb{C})$. Then there is a Zariski open dense subset U in $Hom(\Gamma, GL(n, \mathbb{C}))$ such that, for each $\rho \in U$, $(M_\rho, \mathcal{F}_\rho)$ is both holomorphically and harmonically simple.*

Theorem 5. *Let Λ be a Gromov-hyperbolic group, X the boundary of Λ, and S a compact connected Riemannian manifold with fundamental group Γ. Suppose that Γ acts on X via a homomorphism $\rho : \Gamma \to \Lambda$ and let $(M_\rho, \mathcal{F}_\rho)$ be the corresponding foliated bundle over S. Then $(M_\rho, \mathcal{F}_\rho)$ is harmonically simple. The same holds if Λ is replaced by $SL(2, \mathbb{C})$.*

"Positive" results

Theorem 6. (Holomorphic) *There exists a compact real analytic foliation* $(M, \mathcal{F})$*, a foliated bundle over a compact Riemann surface, and a real analytic leafwise holomorphic function on* M *that is not leafwise constant.*

- $\mathbb{D}$ – unit disc; Γ a cocompact lattice in $SU(1,1)$; $S = \mathbb{D}/\Gamma$;
- $M = (\mathbb{D} \times \mathcal{C})/\Gamma$; $\mathcal{C} = \{[z_1, z_2, t] \in P^4(\mathbb{R}) : |z_1|^2 - |z_2|^2 = t^2\}$;
- $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot [z_1, z_2, t] = [\alpha z_1 + \beta \bar{z}_2, \alpha z_2 + \beta \bar{z}_1, t]$;
- $f(z, [\alpha, \beta, t]) := \frac{\bar{\alpha} z - \beta}{-\bar{\beta} z + \alpha}$. f is Γ-invariant.

Harmonically non-simple, holomorphically simple example

$S = \mathbb{D}/\Gamma$; $\Gamma \subset SU(1,1)$ a cocompact lattice.

Theorem 7. *$\exists$ $(M,\mathcal{F})$, a fol. bund. over S with fiber S^2, such that:*

- *$(M,\mathcal{F})$ is C^ω on complement of pair of leaves, S_1, S_2, homeom. to S;*
- *$(M,\mathcal{F})$ is ergodic with respect to the smooth measure class;*
- *$(S_1 \cup S_2)^c$ has a C^ω compactification, which is an ergodic foliated bundle over S with fiber $S^1 \times [0,2\pi]$;*
- *For both $(M,\mathcal{F})$ and the above analytic compactification, the Liouville property does not hold. A continuous, leafwise harmonic, not leafwise constant, can be found that is real analytic on the complement of $S_1 \cup S_2$.*

A “universal” non-Liouville foliation

- $X_0 = Har(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C}$ harmonic, $|f(z)| \leq 1\}$;
- $PSU(1,1)$ acts on X_0 by $(g, f) \mapsto f \circ g^{-1}$;
- $\Gamma \subset PSU(1,1)$ a cocompact lattice; $M_0 = (\mathbb{D} \times X_0)/\Gamma$;
- $\Phi([z, f]) := f(z)$, $\Phi : M_0 \to \mathbb{C}$.

Finite dimensional examples are obtained by looking for finite dimensional closed orbits of $PSU(1,1)$ on X_0, then restricting Φ to the foliated subspace of M_0 associated to that orbit.

Dynamics of subgroups of $PSU(1,1)$ acting on $Har(\mathbb{D})$

Devaney: A continuous map $f : X \to X$ of a metric space X generates a chaotic dynamical system if:

- There exists a dense orbit (topological transitivity);
- The set of periodic points (finite orbits) is dense;
- Sensitive dependence on initial conditions. (There exists $\delta > 0$ such that for all $x \in X$ and every neighborhood N of x, there exists $y \in N$ and positive integer n such that $f^n(x)$ and $f^n(y)$ are more than δ apart.)

Theorem 8. *Let γ be a hyperbolic or parabolic element of $PSL(2,\mathbb{R})$, regarded as a transformation on $Har(\mathbb{D})$. Then γ defines a chaotic dynamical system.*

Questions

- If $(M, \mathcal{F})$ has codimension 2, is it holomorphically simple?
- If $(M, \mathcal{F})$ has codimension 1, is it harmonically simple?
- Clarify relationship of holom. simple and harm. simple foliations. (Note: if $H^1_{dR}(M, \mathcal{F}) = 0$, harm. simple $\Leftrightarrow$ holom. simple.)
- Let $(M, \mathcal{F})$ be a compact foliated bundle over $\Gamma \backslash G/K$, where G/K is an irreducible locally symmetric space of rank at least two. Show (or give counter-example) that the foliation is harmonically simple.
- Given a hyperbolic and a parabolic element in $PSL(2, \mathbb{R})$, are the dynamical systems they define on $Har(\mathbb{D})$ topologically equivalent?