

Leafwise Holomorphic Functions

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Abstract

It is a well-known and elementary fact that a holomorphic function on a compact complex manifold without boundary is necessarily constant. The purpose of the present article is to investigate whether, or to what extent, a similar property holds in the setting of holomorphically foliated spaces.

1 Introduction and Statement of Results

Suppose that M is a compact manifold, \mathcal{F} is a continuous foliation of M by (not necessarily compact) complex leaves, and that f is a continuous leafwise holomorphic function. The question we wish to study is whether or not f must be leafwise constant.

We will actually work in the setting of *foliated spaces*, as defined in [3]. Thus M is only a topological space, while the leaves of \mathcal{F} admit a smooth manifold structure that varies continuously on M . The term *foliated manifold* will be reserved for when M has a differentiable structure relative to which each leaf of \mathcal{F} is C^1 immersed and the foliation tangent bundle, $T\mathcal{F}$, is a C^0 subbundle of TM . In all situations, it will be assumed that M is compact and connected. We also assume that (M, \mathcal{F}) is a *holomorphically foliated space*, by which we mean that each leaf of \mathcal{F} carries the structure of a complex manifold and that this structure varies continuously on M . The foliated space (M, \mathcal{F}) may, on occasion, also carry a *leafwise Hermitian* metric (resp., is *leafwise Kähler*); that is, the leaves of \mathcal{F} may carry a smooth Hermitian metric (resp., a Kähler metric) that, together with all its derivatives along leaves, varies continuously on M .

If the foliation is such that any continuous leafwise holomorphic function is leafwise constant we will say that (M, \mathcal{F}) – or simply \mathcal{F} – is *holomorphically plain*. It will be proved that a number of general classes of holomorphically foliated manifolds are holomorphically plain. We also give an example of a real analytic, non-plain, holomorphically foliated manifold (with real analytic

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leafwise holomorphic functions that are not leafwise constant) and indicate a general construction that shows how such examples can be obtained.

When the leaves of \mathcal{F} , individually, do not support non-constant bounded holomorphic functions, then clearly \mathcal{F} is holomorphically plain. This is the case, for example, if the universal covering spaces of the leaves are isomorphic as complex manifolds to \mathbb{C}^n . Therefore, the question only becomes meaningful for cases such as, say, a foliation by Riemann surfaces of hyperbolic type, for which the leaves do support non-constant bounded holomorphic functions. It is then necessary to understand the constraining role played by the foliation dynamics.

The subject of this article has a natural counterpart for foliations with leafwise Riemannian metrics and functions that are leafwise harmonic. There is, as well, a discretized form of the problem (of deciding whether leafwise harmonic functions are leafwise constant) in the setting of actions of finitely generated groups and functions that are harmonic along orbits for a combinatorial Laplacian. These “harmonic” variants of the subject contain some essential additional difficulties and, for the sake of keeping this article as elementary as possible, they will be treated elsewhere.

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1.1 Group Actions and Foliated Bundles

Holomorphically foliated spaces arise in a number of ways. For example, as the orbit foliation of a locally free C^1 action of a complex connected Lie group on a compact manifold, or as a foliated bundle over a compact connected complex manifold. See also the work of E. Ghys [4] on laminations by Riemann surfaces and leafwise meromorphic functions.

It is a rather easy fact that if (M, \mathcal{F}) is the orbit foliation of a continuous locally free action of a connected complex Lie group G , then \mathcal{F} is holomorphically plain, where the complex structure on leaves is the one that makes the orbit map $o_x : g \mapsto gx$ (from G onto the leaf of x) a local isomorphism of complex manifolds. We have the following slightly more general fact.

Proposition 1.1 *Let (M, \mathcal{F}) be a holomorphically foliated space such that $T\mathcal{F}$ is holomorphically trivial. Then \mathcal{F} is holomorphically plain.*

The next corollary follows from the proposition and a simple remark due to Wang which is explained later.

Corollary 1.2 *If (M, \mathcal{F}) is a holomorphically foliated space having a dense leaf and $T\mathcal{F}$ is holomorphically trivial, then \mathcal{F} is the orbit foliation of a locally free action of a connected complex Lie group.*

A much more interesting class of examples consists of foliated bundles. We now recall the definition of foliated bundles in the special setting that concerns us here. Let S denote a compact connected complex manifold, let \hat{S} be the

universal covering space of S , and denote by Γ the group of deck transformations of \tilde{S} . Let X be a compact connected space on which Γ acts by homeomorphisms. The action can be represented by a homomorphism $\rho : \Gamma \rightarrow \text{Homeo}(X)$ from Γ into the group of homeomorphisms of X . Then Γ acts on the product $\tilde{S} \times X$ in the following way: $(p, x)\gamma := (p\gamma, \gamma^{-1}(x))$, for $p \in \tilde{S}, x \in X$ and $\gamma \in \Gamma$. Let $M := (\tilde{S} \times X)/\Gamma$ be the space of Γ -orbits. The natural projection $\pi : M \rightarrow S$ gives M the structure of a fiber bundle over S whose fibers are homeomorphic to X , and M is foliated by complex manifolds, transversal to the fibers of π , which are coverings of S . The resulting foliated space will be written $(M_\rho, \mathcal{F}_\rho)$.

Note that the dynamics of a foliated bundle is largely determined by the dynamics of the Γ -action on X . Thus, for example, if X admits a Γ -invariant finite measure of full support (or if the Γ -action has a unique minimal set), then M_ρ admits a completely invariant finite measure of full support (resp., $(M_\rho, \mathcal{F}_\rho)$ has a unique minimal set). It should also be clear that $x \in X$ has a finite Γ -orbit if and only if $\tilde{S} \times \{x\}$ maps to a closed leaf of \mathcal{F}_ρ .

1.2 Leafwise Hermitian and Kähler Foliations

The results in this section assume that (M, \mathcal{F}) is provided with a (continuous on M) leafwise Hermitian metric $\langle \cdot, \cdot \rangle$. We denote by Ω the associated leafwise volume form and define the divergence of a continuous leafwise smooth vector field X to be the function $\text{div}X$ such that $\mathcal{L}_X\Omega = (\text{div}X)\Omega$, where \mathcal{L}_X denotes the Lie derivative along X . A Borel measure m on M is said to be *completely invariant* if

$$m(\text{div}X) := \int_M \text{div}X(x) dm(x) = 0$$

for all continuous, leafwise smooth vector field X . Completely invariant measures are equivalent to *holonomy invariant* transverse measures on (M, \mathcal{F}) (cf. [5]).

A measure m on M will be said to have *full support* if its support coincides with M .

Proposition 1.3 *If (M, \mathcal{F}) is a leafwise Kähler foliated space that admits a completely invariant measure of full support, then \mathcal{F} is holomorphically plain.*

Let $\bar{\partial}^*$ denote the adjoint operator to $\bar{\partial}$ with respect to the chosen leafwise Hermitian metric, and define the $\bar{\partial}$ -Laplacian on leafwise smooth functions by $\Delta_{\bar{\partial}} = \bar{\partial}^*\bar{\partial}$ (we use the notations and sign conventions of [11]). A Borel measure m on M will be called $\Delta_{\bar{\partial}}$ -harmonic, or simply *harmonic*, if

$$m(\Delta_{\bar{\partial}}h) := \int_M (\Delta_{\bar{\partial}}h)(x) dm(x) = 0$$

for all continuous, leafwise smooth function $h : M \rightarrow \mathbb{C}$.

Proposition 1.3 is an immediate consequence of the next proposition, itself an immediate consequence of [6, Theorem 1(b)]. (We remark that on a Kähler manifold $\Delta_{\bar{\partial}}h = -\frac{1}{2}\text{div grad}h$.)

Proposition 1.4 *Suppose that (M, \mathcal{F}) is a leafwise Hermitian foliated space. Also suppose that the union of the supports of all $\Delta_{\bar{\partial}}$ -harmonic measures is M . Then (M, \mathcal{F}) is holomorphically plain.*

1.3 Foliations with Few Minimal Sets

Unless specified otherwise, (M, \mathcal{F}) will continue to denote a compact, connected, holomorphically foliated space. We recall that a *minimal set* X of (M, \mathcal{F}) is a closed, non-empty, \mathcal{F} -saturated subset of M that has no proper subset with these same properties. If M is, itself, a minimal set, then \mathcal{F} is said to be a *minimal foliation*.

Proposition 1.5 *Suppose that the closure of each leaf of (M, \mathcal{F}) contains (at most) countably many minimal sets. Then \mathcal{F} is holomorphically plain.*

Proposition 1.5 clearly applies to minimal foliations. Also note that if \mathcal{F} is a (transversely) Riemannian foliation, then the closure of each leaf is a minimal set [13], so the proposition also applies. Therefore one has the next corollary, which will be used later a number of times. (Note that a compact group of diffeomorphisms of a compact manifold must preserve a Riemannian metric.)

Corollary 1.6 *Let $(M_\rho, \mathcal{F}_\rho)$ be a holomorphically foliated bundle over S with fiber a compact manifold X , where the homomorphism ρ has values in a compact group of diffeomorphisms of X . Then \mathcal{F}_ρ is holomorphically plain.*

There is also a large class of foliated bundles associated to Γ -actions on projective space $\mathbb{F}P^n$, $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and derived from linear representations of Γ , for which the hypothesis of the proposition are satisfied. This will be described after introducing some notations. Let $GL(W)$ be the group of linear automorphisms of W , where W is a vector space over \mathbb{F} . The quotient of $GL(W)$ by its center will be written $PGL(W)$ and the projective space associated to W will be written $P(W)$. If $W = \mathbb{F}^n$, we write $PGL(W) = PGL(n, \mathbb{F})$ and $P(W) = \mathbb{F}P^{n-1}$.

Let $\rho : \Gamma \rightarrow GL(W)$ be a linear representation of a group Γ on a vector space W . An element $\gamma \in \Gamma$ will be called *proximal* if the maximal characteristic exponent of $\rho(\gamma)$ is simple. The next result is a consequence of Proposition 1.10 and [12, 3.4 and 3.6, Chapter VI].

Proposition 1.7 *Let S be a connected, compact, complex manifold with fundamental group Γ , W an n -dimensional vector space over \mathbb{F} , and $\rho : \Gamma \rightarrow GL(W)$ a continuous homomorphism for which Γ contains a proximal element. Let $(M_\rho, \mathcal{F}_\rho)$ be the foliated bundle over S with fiber $P(W)$, where Γ acts on $P(W)$ via ρ . Then $(M_\rho, \mathcal{F}_\rho)$ is holomorphically plain.*

The hypothesis of Proposition 1.7 holds if the image of Γ in $PGL(W)$ is Zariski dense and not precompact [12, Theorem 4.3(i)]. If the image is precompact, we can apply Corollary 1.6, so the following corollary holds.

Corollary 1.8 *Let S be a connected, compact, complex manifold with fundamental group Γ , W an n -dimensional vector space over \mathbb{F} , and $\rho : \Gamma \rightarrow GL(W)$ a continuous homomorphism such that the image of Γ in $PGL(W)$ is Zariski dense. Let $(M_\rho, \mathcal{F}_\rho)$ be the foliated bundle over S with fiber $P(W)$, where Γ acts on $P(W)$ via ρ . Then $(M_\rho, \mathcal{F}_\rho)$ is holomorphically plain.*

It will follow from Corollary 1.8 that foliated bundles associated to projective (linear) actions of $\pi_1(S)$, for a compact Riemann surface S , are generically holomorphically plain, in the sense described below.

We first recall some definitions. Let $S = \mathbb{D}/\Gamma$ be a surface of genus $g \geq 2$, where \mathbb{D} denotes the Poincaré disc and Γ is a cocompact discrete group (without torsion) of hyperbolic isometries. Let G be an algebraic group and denote by $\text{Hom}(\Gamma, G)$ the variety of homomorphisms from Γ to G . The structure of algebraic variety is obtained by identifying $\text{Hom}(\Gamma, G)$ with a subvariety of G^{2g} defined by equations representing relations among elements in a generating set for Γ .

Theorem 1.9 *Let $G = GL(n, \mathbb{C})$. Then there is a Zariski open dense subset U in $\text{Hom}(\Gamma, G)$ such that, for each $\rho \in U$, the foliated bundle $(M_\rho, \mathcal{F}_\rho)$ for the corresponding Γ -action on $\mathbb{C}P^{n-1}$ is holomorphically plain.*

The key step in the results mentioned in this section is the proposition given next. A topological space X equipped with an action of a group Γ by homeomorphisms will be called here a *convergence Γ -space*, or a Γ -space of *convergence type*, if the following holds: there exists a (at most) countable family of subsets $X_i \subset X$, $i = 1, 2, \dots$, such that (i) the intersection of all the X_i is (at most) countable and (ii) for each i there is a sequence $\gamma_m \in \Gamma$ and a point $x_i \in X$ such that $\gamma_m(y)$ converges to x_i as $m \rightarrow \infty$, for each y in the complement of X_i .

As a simple example, let $X = \mathbb{C}P^1$ and $\rho : \Gamma \rightarrow PSL(2, \mathbb{C})$ a homomorphism such that $\rho(\Gamma)$ is not relatively compact. Then X , with the Γ -action obtained from ρ , is a convergence Γ -space. The Γ -actions on $\mathbb{F}P^{n-1}$ of Proposition 1.7 as well as the natural action of any unbounded subgroup Γ of a Gromov-hyperbolic group G on the boundary ∂G , also define convergence Γ -spaces.

Proposition 1.10 *Let S be a compact complex manifold with fundamental group Γ , let X be a compact Γ -space of convergence type, and let (M, \mathcal{F}) be the corresponding foliated bundle over S . Then \mathcal{F} is holomorphically plain.*

A class of Γ -spaces for which the convergence property is well known to hold consists of actions of non relatively compact subgroups of a Gromov-hyperbolic group on the boundary of the latter. (See, for example, [7] and references cited there. It should be noted that the standard definition of the convergence property used in the literature on hyperbolic groups is much more restrictive than the one we are using here.) Therefore (keeping in mind Corollary 1.6, the following holds.

Proposition 1.11 *Let G be a Gromov-hyperbolic group, X the boundary of G , and S a compact connected complex manifold with fundamental group Γ . Suppose that Γ acts on X via a homomorphism $\rho : \Gamma \rightarrow G$ and let $(M_\rho, \mathcal{F}_\rho)$ be the corresponding foliated bundle over S . Then \mathcal{F}_ρ is holomorphically plain.*

Corollary 1.12 *Let S be a compact Riemann surface, let $\rho : \Gamma \rightarrow G$ be a homomorphism of the fundamental group of S into a connected simple Lie group of rank one, and let $(M_\rho, \mathcal{F}_\rho)$ be the foliated bundle over S with fibers X , where X is the boundary at infinity of the Riemannian symmetric space associated to G . Then \mathcal{F}_ρ is holomorphically plain.*

1.4 Codimension-One

The main idea used in the proof of Proposition 1.5, together with elementary facts about the structure of codimension-one foliations yield the following.

Theorem 1.13 *If (M, \mathcal{F}) has codimension 1, then it is holomorphically plain.*

1.5 An Example

The results described so far might lead one to expect that holomorphically foliated spaces are holomorphically plain under very general conditions and that one should be able to prove it using only qualitative properties of leafwise holomorphic functions. The next theorem shows, however, that the situation cannot be so simple.

Theorem 1.14 *There exists a compact real analytic foliation (M, \mathcal{F}) , which is a foliated bundle over a compact Riemann surface, and a real analytic leafwise holomorphic function $f : M \rightarrow \mathbb{C}$ that is not leafwise constant.*

To construct an example of (M, \mathcal{F}) and f as in Theorem 1.14 we first introduce some notation. Let \mathbb{D} denote as before the unit open disk in \mathbb{C} . Points in projective space $\mathbb{R}P^4$ will be written $[z_1, z_2, t]$, where $z_i \in \mathbb{C}, t \in \mathbb{R}$ and (z_1, z_2, t) is non-zero. Define a (real analytic) action of $SU(1, 1)$ on $\mathbb{R}P^4$ as follows. Elements of $SU(1, 1)$ are matrices of the form $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ for which $|\alpha|^2 - |\beta|^2 = 1$. The action of $SU(1, 1)$ on $\mathbb{R}P^4$ defined by

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot [z_1, z_2, t] := [\alpha z_1 + \beta \bar{z}_2, \alpha z_2 + \beta \bar{z}_1, t]$$

leaves invariant the submanifold $\mathcal{C} := \{[z_1, z_2, t] \in \mathbb{R}P^4 : |z_1|^2 - |z_2|^2 = t^2\}$.

We define on $\mathbb{D} \times \mathcal{C}$ the function

$$f(z, [\alpha, \beta, t]) := \frac{\bar{\alpha}z - \beta}{-\bar{\beta}z + \alpha}.$$

An elementary calculation shows that $f(gz, g[\alpha, \beta, t]) = f(z, [\alpha, \beta, t])$ for every $g \in SU(1, 1)$. Therefore, if Γ is a uniform lattice in $SU(1, 1)$, then f yields a function on the foliated bundle $M = (\mathbb{D} \times \mathcal{C})/\Gamma$ that is real analytic, leafwise holomorphic and the restriction of f to any leaf for which $t \neq 0$ is not constant.

1.6 A Universal Non-Plain Foliated Space

We describe now a kind of “universal space” from which such examples can be constructed. This will be done in the setting of foliated bundles whose base are compact Riemann surfaces, but it should be apparent that the same ideas apply more broadly. Let $X_0 := \text{Hol}(\mathbb{D}, \overline{\mathbb{D}})$ be the space of holomorphic functions defined on \mathbb{D} such that $\sup\{|f(z)| : z \in \mathbb{D}\} \leq 1$. Then X_0 , with the topology of uniform convergence on compact sets, is a compact metrizable space upon which $PSU(1, 1)$ acts via the continuous action: $(g, f) \mapsto f \circ g^{-1}$, where g , on the right-hand side, is regarded as an automorphism (a Möbius transformation) of the Poincaré disc.

Let now $\rho : \Gamma \rightarrow PSU(1, 1)$ be a homomorphism from the fundamental group of a compact Riemann surface $S = \mathbb{D}/\Gamma$ into the Möbius group, and construct the foliated bundle $(\overline{S} \times X_0)/\Gamma$ over S . We will denote the resulting foliated space by (M_0, \mathcal{F}_0) . By a *morphism* $f : (M, \mathcal{F}) \rightarrow (M_0, \mathcal{F}_0)$ we will mean a (continuous) $f : M \rightarrow M_0$ that maps leaves to leaves holomorphically such that $\pi_{M_0} \circ f = \pi_M$, where π_M (resp., π_{M_0}) is the natural projection from M to S (resp., from M_0 to S).

A leafwise nonconstant, leafwise holomorphic function can now be produced on (M_0, \mathcal{F}_0) by the following essentially tautological procedure. First define $\bar{\phi} : \mathbb{D} \times X_0 \rightarrow \mathbb{C}$ by $\bar{\phi}(z, f) := f(z)$. Note that $\bar{\phi}(\gamma(z), f \circ \gamma^{-1}) = \bar{\phi}(z, f)$ for each $\gamma \in \Gamma$. There is as a result a well-defined function $\phi : M_0 \rightarrow \mathbb{C}$ such that $\phi \circ \pi = \bar{\phi}$, where π is the natural projection from $\mathbb{D} \times X \rightarrow M_0$. The function $\phi : M_0 \rightarrow \mathbb{C}$ is a continuous, leafwise holomorphic function.

The following remark is an immediate consequence of these definitions. In the proposition, equivariance of a map $\hat{\psi} : V \rightarrow \mathbb{C}$ means that $\hat{\psi} \circ \gamma = \gamma \circ \hat{\psi}$ for each $\gamma \in \Gamma$.

Proposition 1.15 *Let (M, \mathcal{F}) be a foliated bundle over $\Gamma \backslash \mathbb{D}$ with fiber V . Then there is a one-to-one correspondence between (continuous) leafwise holomorphic functions $\psi : M \rightarrow \mathbb{C}$ and Γ -equivariant (continuous) $\hat{\psi} : V \rightarrow X_0$. Furthermore, if $\Psi : M \rightarrow M_0$ is the morphism of holomorphically foliated spaces induced from $\hat{\psi}$, then $\psi = \phi \circ \Psi$, and Ψ is the unique morphism from M to M_0 that satisfies this last equality.*

The proposition indicates how to go about looking for examples of foliated manifolds that are not holomorphically plain: one tries to find a Γ -invariant manifold V embedded in X . Specifically, one can try to obtain a manifold $V \subset X$ as the closure of a $PSU(1, 1)$ -orbit. In fact, the example given just after Theorem 1.14 is closely related to what one gets by taking V to be the closure of the $PSU(1, 1)$ -orbit of the function $\varphi(z) = z$ in $\text{Hol}(\mathbb{D}, \overline{\mathbb{D}})$. This closure is the compactification of a 3-manifold (an open solid torus) by adding a circle at infinity (so as to form a 3-sphere), while the submanifold $\mathcal{C} \subset \mathbb{R}P^4$ of that example corresponds to an analytic “doubling” of this 3-manifold.

The remark just made suggests that a precise characterization of holomorphically plain foliated bundles over a compact Riemann surface will require an investigation of the dynamics of the action of $PSU(1, 1)$ on $\text{Hol}(\mathbb{D}, \overline{\mathbb{D}})$. Such a

characterization should tell, in particular, how common holomorphically plain foliations are, at least in this special setting.

2 Proofs

2.1 Proposition 1.1 and Corollary 1.2

The hypothesis that $T\mathcal{F}$ is holomorphically trivial means the following: there exist vector fields, X_1, \dots, X_l , on M , where l is the leaf dimension, such that the X_i are everywhere tangent to \mathcal{F} , linearly independent, and define holomorphic vector fields on leaves. Furthermore, the X_i , together with their tangential derivatives of first order, are continuous on M .

Since M is compact, the X_i are complete vector fields and each flow line is the image of a holomorphic map from \mathbb{C} into a leaf. Therefore, the restriction to orbits of leafwise holomorphic functions define bounded holomorphic functions on \mathbb{C} . As a result, such functions are constant on orbits, hence leafwise constant.

To show the corollary, write $[X_i, X_j] = \sum_k f_{ij}^k X_k$. The coefficients f_{ij}^k are continuous, leafwise holomorphic, hence leafwise constant. Due to the existence of a dense leaf, these coefficients are constant on M . Therefore, the X_i span a finite dimensional Lie algebra. The corollary is now a result of standard facts in Lie theory.

2.2 Proposition 1.4

A leafwise holomorphic function f is $\Delta_{\bar{\partial}}$ -harmonic since $\bar{\partial}f = 0$, so the proposition is an immediate result of [6, Theorem 1(b)]. We give here an independent elementary proof when (M, \mathcal{F}) is leafwise Kähler. It is well known (see, for example, [8, p. 115]) that under this extra hypothesis $\Delta_{\bar{\partial}}\bar{f} = \Delta_{\partial}\bar{f} = 0$, so we also have $\Delta_{\bar{\partial}}\bar{f} = 0$. Another elementary calculation gives the identity

$$\Delta_{\bar{\partial}}(|f|^2) = -\|\partial f\|^2 + f\Delta_{\bar{\partial}}\bar{f} = -\|\partial f\|^2$$

holds. By integrating the resulting equality against a harmonic measure m one deduces that $\partial f = 0$ m -almost everywhere, and as ∂f is continuous, it is zero on the support of m . Therefore, f must be constant on that support.

2.3 Proposition 1.5

The next lemma and corollary will be needed a number of times in the rest of the paper.

Lemma 2.1 *Let (M, \mathcal{F}) be a holomorphically foliated compact connected space and let $f : M \rightarrow \mathbb{C}$ be a leafwise holomorphic function. Let $C \subset M$ be the set where f is leafwise constant. Suppose that $f(C) \subset \mathbb{C}$ is at most countable. Then f is constant on M .*

Proof. Clearly C is a compact \mathcal{F} -saturated set. By the open mapping theorem for holomorphic functions, the restriction of f to each leaf in the complement of C is open, hence f itself is an open mapping on that complement. Therefore $U := f(M \setminus C)$ is a bounded open subset of \mathbb{C} . Since M is compact, $f(M) = U \cup f(C)$ is compact. In particular, the boundary of U is contained in $f(C)$, which is by assumption a countable set. But a bounded non-empty open set in \mathbb{C} cannot have a countable boundary. Therefore U is empty and (as M is connected) $f(M)$ reduces to a point. \square

Corollary 2.2 *Suppose that (M, \mathcal{F}) has (at most) countably many minimal sets. Then any leafwise holomorphic function is constant on M .*

Proof. Let f be a leafwise holomorphic function and C as in Lemma 2.1. Denote by \mathcal{M} the union of minimal sets. We claim that $f(C) = f(\mathcal{M})$. Indeed if $x \in C$ and $f(x) = c$, then f takes the constant value c on the closure of the leaf containing x . But this closure contains a minimal set, so $f(C) \subset f(\mathcal{M})$. Conversely, by an application of the maximal principle for holomorphic functions, each closed saturated set contains leaves where f is constant, so $f(C) = f(\mathcal{M})$ as claimed. The main assertion is now a consequence of Lemma 2.1. \square

By applying Corollary 2.2 to the closure of each leaf we obtain Proposition 1.5.

2.4 Proposition 1.9

By [14], $\text{Hom}(\Gamma, G)$ is irreducible, and by [1, 8.2] the homomorphisms with Zariski dense image form a Zariski open subset $\mathcal{D} \subset \text{Hom}(\Gamma, G)$. On the other hand, \mathcal{D} is clearly nonempty. (Γ has homomorphisms onto the free group on 2 generators and it is easy to show that the free group has representations with Zariski dense image.) Therefore, the theorem is a consequence of Corollary 1.8.

2.5 Corollary 1.10

We identify X with a fixed fiber of the foliated bundle, so that the holonomy transformations of \mathcal{F} correspond to the Γ -action on X .

Suppose that f is a leafwise holomorphic continuous function on M . Let $C \subset X$ be the compact Γ -invariant subset that corresponds to leaves on which f is constant. Then there is a countable set of points x_1, x_2, \dots such that for all $y \in C$, with the possible exception of a countable subset of C , we can find a sequence γ_m such that $\gamma_m x \rightarrow x_i$ for some i . Clearly the x_i belong to C . Therefore f can take at most a countable set of values on leaves of \mathcal{F} in C . So f must be leafwise constant by Lemma 2.1.

2.6 Theorem 1.13

The basic facts about codimension-one foliations that we will use can be found in [10] or [3], for example. We recall that a minimal set is said to be *exceptional* if it is neither a single closed leaf of (a codimension-one foliation) \mathcal{F} nor a connected component of M .

By a theorem of Haefliger [9] (see also [10]), the union of compact leaves of \mathcal{F} is a compact set, which we denote by N . Let L be a connected component of the complement of N . Then, as \mathcal{F} has codimension 1, the closure \bar{L} is a compact manifold whose boundary is a finite union of compact leaves (cf. [10]).

It is known that (M, \mathcal{F}) has only a finite set of exceptional minimal sets. In particular, $(\bar{L}, \mathcal{F}|_{\bar{L}})$ has finitely many minimal sets (exceptional or not). Applying Corollary 2.2 we deduce that any leafwise holomorphic function is constant on \bar{L} . In particular, any leafwise holomorphic function is leafwise constant on $M \setminus N$. The same is obviously true on N (which is the union of closed leaves). Therefore (M, \mathcal{F}) is holomorphically plain.

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