1

A DIFFERENTIAL-GEOMETRIC VIEW OF NORMAL FORMS OF CONTRACTIONS

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ABSTRACT. A version of non-stationary normal forms theory was recently obtained by M. Guysinsky and A. Katok, which has become an important ingredient in several recent investigations of rigidity of group actions. Our main aim is to explain their results, the *sub-resonance normal forms* and *centralizer* theorems, from a differential-geometric perspective.

1. INTRODUCTION

The theory of normal forms of dynamical systems has a long and venerable history, going back to Poincaré's study of analytic differential equations near a singular point. Let $\dot{x} = Ax + \cdots$ be a system of such equations, where A is an n-by-n matrix. One of Poincaré's central results on the subject is that it is possible to find a formal change of coordinates $x = y + \cdots$ that brings the original equation to linear form $\dot{y} = Ay$ under the hypotheses that the eigenvalues of A do not satisfy certain arithmetic relations called *resonances*.

If resonance relations do exist, it may not be possible to linearize the equations in the same way, but further work by Poincaré and Dulac showed that it is still possible to bring them to polynomial form. Sternberg [St] later obtained such normal forms (linearization or polynomial form in the presence of resonances) for non-analytic equations, when the associated flow defines a contracting dynamical system. The simplified form of the flow in this case holds not only infinitesimally to infinite order but on a neighborhood of the singular point, and the coordinate change is C^∞.

Non-stationary linearization and normal forms have to do with the same type of simplification of dynamical systems, now along an arbitrary orbit, not necessarily a fixed point. It was studied by, among others, Sibuya, Kostin, Yomdin. (See [Do] and the references cited there.)

A recent non-stationary normal forms theorem for contractions has been derived in [GK] by M. Guysinsky and A. Katok. The main results in [GK] are a *sub-resonance normal form theorem*, which obtains, under a key *narrow band spectrum* hypotheses, normal forms for families of smooth contractions; and a *centralizer theorem*, which shows that smooth mappings commuting with the contracting map (but not necessarily contractions themselves) automatically assume a similar normal form.

The paper by Guysinsky and Katok has been used as an essential tool in several recent studies of so-called *rigidity of group actions*. Rigidity theorems are of several kinds but, broadly, they assert that certain classes of group actions are isomorphic in some appropriate sense (C^∞ conjugate, measurably equivalent, etc.) to model actions obtained by essentially algebraic constructions. Papers in which normal forms are used explicitly (typically having to do with the special case of linearization) are variously concerned with rigidity of actions of: higher-rank abelian groups $\mathbb{R}^k$ and $\mathbb{Z}^k$, $k \geq 2$, as in [KK1, KK2, KKS, KNT, NT] for example; semisimple Lie groups of higher rank and their lattice subgroups, as in [Hu, KL1, KL2, KLZ, KSp, MQ, FM]; and actions of Anosov diffeomorphisms and flows, as in [Sa], for a recent example. In a less explicit way, non-stationary linearization is used

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under geometric guise in [BFL, BL, Fe1, Fe2, FL], to cite a small and probably biased sample. (For a broad discussion of rigidity of group actions, see [FK].)

The present article resulted from my effort to understand [GK]. In the abstract of their paper, Guysinsky and Katok remark: "One of the ways to formulate our first conclusion (the sub-resonance normal form theorem) is to say that there is a continuous invariant family of geometric structures in the fibers whose automorphism groups are certain finite-dimensional Lie groups." The precise nature of this geometric structure, however, is not made explicit there in a way that looks most natural from a differential geometric viewpoint. It would, perhaps, be most satisfying to derive, as a first step toward the theorem, an invariant infinitesimal structure such as a connection on a principal bundle, a Cartan structure of finite type, or at least some rigid structure in the sense of Gromov (structures having finite dimensional automorphism groups); and then proceed to obtain the conclusion of the theorem by integration, or development. This is the outlook on the theory that we take here.

The geometric viewpoint can be illustrated in clearest form for a contracting automorphism of a foliation in a compact manifold under the *half-pinching* assumption, a special case of the narrow-band spectrum hypotheses introduced in [GK]. The infinitesimal structure in this case is an ordinary connection, or covariant derivative of vector fields; the coordinate changes bringing the contraction to normal form are directly related to the exponential map associated to the connection. (This is studied in [Fe1] in the context of Anosov diffeomorphisms with smooth foliations. An application to the study of actions of lattices in higher-rank semisimple Lie groups is obtained in [Fe2].)

Under the hypotheses of [GK], such ideal formulation is not generally possible due to lack of differentiability of an invariant flag of subbundles that will be encountered shortly. (This flag is trivial, hence smooth, under the half-pinched spectrum hypotheses.) Nevertheless, what is obtained does resemble, in a formal sense, geometric structures of finite type. When differentiability is available, we do obtain Gromov-rigid structures. They are, in fact, generalized flat connections of higher order. The definitions and precise statements are given in the next section.

We make no attempt to state results in their most general form. The setting of continuous laminations of compact metric spaces, for example, which is adopted here in place of fiber bundles over compact metric spaces, as in [GK], is more restrictive than necessary, but it is the setting in which the theorems are generally used. The contraction map is assumed here to be C^∞ on the leaves of the lamination. The reader interested in more refined analytic statements should consult [Gu] and [Do].

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2. THE NON-STATIONARY NORMAL FORMS THEOREM

The general setting of [GK] consists of a family of manifolds $\mathcal{F}(x)$, where x ranges over a compact metric space M, and a family of contracting diffeomorphisms $f : \mathcal{F}(x) \to \mathcal{F}(\underline{f}x)$ associated to a homeomorphism $\underline{f}$ of M. The objective is to obtain a family of smooth coordinates on the $\mathcal{F}(x)$, depending continuously on x, under which f assumes polynomial form. A somewhat more restrictive setting is actually used here: $\mathcal{F}$ will correspond to a continuous lamination of a compact metric space, M, having smooth leaves, and a lamination-preserving, leaf contracting homeomorphism of M, smooth on leaves. This is not an essential restriction but we adopt it for convenience. (It makes sense in this case to write $\underline{f} = f$.) The prime example is an Anosov diffeomorphism of a compact manifold, in which case $\mathcal{F}$ is the stable Anosov foliation.

For the rest of the section, let $(M,\mathcal{F})$ be such a lamination. A homeomorphism f of M that maps leaves to leaves diffeomorphically will be called a *smooth automorphism* of $\mathcal{F}$. The group of smooth automorphisms will be written $\mathrm{Aut}(M,\mathcal{F})$. By $T_x\mathcal{F}$ we denote the tangent space to the leaf $\mathcal{F}(x)$ at x. The tangent bundle, $T\mathcal{F}$, of $\mathcal{F}$ is the union of these spaces. The derivative map on $T\mathcal{F}$ induced by f is written $df_x : T_x\mathcal{F} \to T_{f(x)}\mathcal{F}$. An automorphism $f \in \mathrm{Aut}(M,\mathcal{F})$ is *contracting* if, for some k, the norm $\|df_x^k\|$ is bounded by a constant $C<1$. This definition does not depend on the choice of Riemannian metric. For simplicity of notation, it will be assumed that $k=1$. (If the normal forms theorem holds for f^k, for some $k\geq 1$, it necessarily also holds for f, due to the part of the theorem that deals with the centralizer of the contraction.)

Throughout the paper, we say that a given structure is 'smooth' (or C^k) if its restriction to leaves of $\mathcal{F}$ is smooth. Of course, differentiability does not make sense on directions that are transverse to the leaves.

2.1. **The Main Theorems.** First observe that the leaves of $\mathcal{F}$ are diffeomorphic to $\mathbb{R}^n$. In fact, for each $x\in M$, $\mathcal{F}(x)$ can be written as a union of nested sets of the form $f^{-k}(B_a(f^k(x)))$, where $B_a(y)\subset\mathcal{F}(y)$ is a ball of radius a centered at y. Let $F(M,\mathcal{F})_x$ be the set of all smooth diffeomorphisms from $\mathbb{R}^n$ to $\mathcal{F}(x)$ sending 0 to x and define $F(M,\mathcal{F})$ as the disjoint union of the $F(M,\mathcal{F})_x$, for $x\in M$. Let $\mathcal{D}(\mathbb{R}^n)_0$ denote the group of diffeomorphisms of $\mathbb{R}^n$ that fix 0. We give both $F(M,\mathcal{F})$ and $\mathcal{D}(\mathbb{R}^n)_0$ the C^∞-topology. More precisely, a sequence $\varphi_m\in F(M,\mathcal{F})$ converges to $\varphi\in F(M,\mathcal{F})$ if for each compact $K\subset\mathbb{R}^n$ and each integer $k\geq 0$, the k-jet of φ_m converges uniformly on K to the k-jet of φ.

There is a natural continuous action of $\mathcal{D}(\mathbb{R}^n)_0$ on $F(M,\mathcal{F})$ by right-composition: $(\sigma,\varphi)\mapsto\sigma\circ\varphi$, for $(\sigma,\varphi)\in F(M,\mathcal{F})\times\mathcal{D}(\mathbb{R}^n)_0$. We regard $F(M,\mathcal{F})$ as a principal bundle over M with structure group $\mathcal{D}(\mathbb{R}^n)_0$. The group $\mathrm{Aut}(M,\mathcal{F})$ acts on $F(M,\mathcal{F})$ by principal bundle automorphisms, via left-composition: $(f,\sigma)\mapsto f\circ\sigma$, for f in $\mathrm{Aut}(M,\mathcal{F})$ and σ in $F(M,\mathcal{F})$.

We are interested in obtaining continuous invariant *reductions* of $F(M,\mathcal{F})$. This means the following. Let H be a subgroup of $\mathcal{D}(\mathbb{R}^n)_0$ and $P\subset F(M,\mathcal{F})$ a principal subbundle with structure group H for the same right-action by composition of maps. We say in this case that P is an H-reduction of $F(M,\mathcal{F})$. Invariance under $f\in\mathrm{Aut}(M,\mathcal{F})$ means that the set $f(P)$ coincides with P. Observe that if $\sigma\in P_x$ and $\sigma'\in P_{f(x)}$, then $f\circ\sigma=\sigma'\circ\phi$ for some $\phi\in H$.

The non-stationary normal forms (resp., linearization) theorem is fundamentally the assertion that an f-invariant H-reduction of $F(M,\mathcal{F})$ exists, for a contraction f and a subgroup H of $\mathcal{D}(\mathbb{R}^n)_0$ consisting of polynomial (resp., linear) maps of $\mathbb{R}^n$. The group H will be written $H_{\mathcal{V}}$, where the symbol $\mathcal{V}$, to be defined shortly, stands for data about the spectrum of the operator that f induces on vector fields. The details will be spelled out in Section 3.

Let X be a vector field on $\mathcal{F}$. The *push-forward* of X under f is the vector field f_*X on $\mathcal{F}$ defined by $(f_*X)(x):=df_{f^{-1}(x)}X(f^{-1}(x))$, $x\in M$. This gives a linear operator on the Banach space $\Gamma^0(T\mathcal{F})$ of continuous vector fields on $T\mathcal{F}$. The spectrum of the complexification of f_* is the so-called *Mather spectrum* of f. A contraction f is said to satisfy the *narrow band spectrum* condition if the Mather spectrum of f has the properties:

(1) There are numbers a_i, b_i, for $i=1,\cdots,l$ (where l is at most the dimension of $\mathcal{F}$) such that $a_1\leq b_1<\cdots<a_l\leq b_l<0$ and $b_i-a_i<-b_l$ for all i;
(2) the Mather spectrum of f is contained in the union of closed rings, $\mathcal{R}_i\subset\mathbb{C}$, of radii e^{a_i}, e^{b_i}, $i=1,\cdots,l$;

The above assumptions imply that the invariant subspace $\mathcal{E}_i\subset\Gamma^0(T\mathcal{F})$ associated to $\mathcal{R}_i$ is a module over the ring of continuous functions; the disjoint union of subspaces $E_i(x):=\{X(x)\in T_x\mathcal{F}\mid X\in\mathcal{E}_i\}$

constitutes an f-invariant continuous subbundle of $T\mathcal{F}$, and $T\mathcal{F}$ decomposes as a direct sum:

$$T\mathcal{F} = E_1 \oplus \cdots \oplus E_l. \tag{1}$$

The spectrum of the restriction of f_* to the continuous sections of E_i lies in $\mathcal{R}_i$.

Denote $V_i := \mathbb{R}^{n_i}$, where n_i is the (fiber) dimension of E_i and $V := \mathbb{R}^n$. Then V decomposes as direct sum of the V_i, where n is the (leaf) dimension of $\mathcal{F}$. We define $I_i = [a_i, b_i]$ and

$$\mathcal{V} := \{(V_i, I_i) \mid i = 1, \ldots, l\}. \tag{2}$$

$\mathcal{V}$ will be called a *spectral partition* of V associated to f. $\mathcal{V}$ will be said to satisfy the *narrow band spectrum* condition if the length of each interval I_i is less than the spectral gap; that is, for each $i = 1, \cdots, l$,

$$b_i - a_i < -b_l. \tag{3}$$

Define, inductively, the numbers $\lambda_1, \cdots, \lambda_l$ by $\lambda_l = a_l$ and $\lambda_{i-1} = \lambda_i a_{i-1}/b_i$, $i = l, l-1, \cdots, 2$. Note that $\lambda_1 < \lambda_2 < \cdots < \lambda_l$. Also define $d_{\mathcal{V}}$ as the integer part of λ_1/λ_l and $T_{\mathcal{V}} : V \to V$ as the linear isomorphism such that $T_{\mathcal{V}} u = e^{\lambda_i} u$, for $u \in V_i$.

The structure group, $H_{\mathcal{V}}$, of the above mentioned reductions is now defined as the set of polynomial mappings in $\mathcal{D}(\mathbb{R}^n)_0$ that are $T_{\mathcal{V}}$*-stable*, i.e., such that $T^n_{\mathcal{V}} \circ h \circ T^{-n}_{\mathcal{V}}$ is bounded over compacts sets for all n. It will be seen later that this is a Lie group of polynomial maps of degree bounded above by $d_{\mathcal{V}}$. We refer to $H_{\mathcal{V}}$ as the $\mathcal{V}$*-stable group*.

The main results of [GK] can now be stated as follows.

Theorem 1. *Let $G \subset Aut(M, \mathcal{F})$ be a group whose center contains a contracting element with spectral partition $\mathcal{V}$ satisfying the narrow band spectrum condition. Then $F(M, \mathcal{F})$ admits a unique continuous G-invariant $H_{\mathcal{V}}$-reduction.*

In general, the decomposition of $T\mathcal{F}$ into the subbundles E_i is only continuous except, of course, when the decomposition is trivial, i.e., $l = 1$. It is interesting to know how the degree of smoothness of the reduction depends on the degree of smoothness of the E_i. It turns out that they are the same. More precisely, the following theorem holds. (See [Gu], Theorem 3.) We refer to the decomposition $T\mathcal{F} = E_1 \oplus \cdots \oplus E_l$ as the $\mathcal{V}$*-splitting of* $T\mathcal{F}$ *associated to* f. (Observe that Theorem 2 presupposes that an appropriate C^k-manifold structure has been defined on the leaves of $F(M, \mathcal{V})$, making them infinite dimensional manifolds. This is obtained implicitly in [Gu]. Since our discussion will be limited to the aspects of the theory that depend only on jet spaces of finite order, we will not discuss this point further here.)

Theorem 2. *Let f be a contraction of $(M, \mathcal{F})$ and $\mathcal{V}$ an associated spectral partition satisfying the narrow band spectrum condition, Suppose that the $\mathcal{V}$-splitting of $T\mathcal{F}$ is C^k. Then, the unique f-invariant $H_{\mathcal{V}}$-reduction of $F(M, \mathcal{V})$ is C^k along leaves.*

These theorems can be rephrased to resemble more closely the statements of [GK]. We first need to define the notion of *polynomial maps of sub-resonance type*. A remark about terminology must be made at this point. To be in strict agreement with [GK], these maps should be called *sub-resonance generated* polynomial maps, but the distinction is not of much consequence here and we omit the extra word.

Fix $x, y \in M$, and non-negative integers $j, i_1, \ldots, i_r$, for some $r \geq 0$. Write $I = (i_1, \ldots, i_r)$ and define $E^j_I(x,y) := E^*_{i_1}(x) \otimes \cdots \otimes E^*_{i_r}(x) \otimes E_j(y)$. A homogeneous polynomial map $F : T_x\mathcal{F} \to T_y\mathcal{F}$ of degree r uniquely decomposes as a sum of terms $F^j_I \in E^j_I(x,y)$. Thus, the space $T^{(r,1)}\mathcal{F}$ of tensors over $T\mathcal{F}$ of type $(r, 1)$, decomposes as direct sum of the f-invariant subspaces E^j_I.

A polynomial map from $T_x\mathcal{F}$ to $T_y\mathcal{F}$ will be said to be of *sub-resonance type* if for each r and each of its homogeneous components of order r, a factor F_I^j is nonzero only if

$$\lambda_j \leq \lambda_{i_1} + \cdots + \lambda_{i_r}. \tag{4}$$

These are polynomial maps of degree no greater than $d_{\mathcal{V}}$. We say that an element of $T^{(r,1)}\mathcal{F}$ is a tensor of *sub-resonance type* if it lies in the linear span of the E_I^j for which $\lambda_j \leq \lambda_{i_1} + \cdots + \lambda_{i_r}$.

Theorem 3. *Let f be a smooth contracting automorphism of $(M,\mathcal{F})$, where M is a compact metric space and $\mathcal{F}$ a continuous lamination of M with smooth leaves. Suppose that f satisfies the narrow band spectrum condition. Then, there exists a continuous family of smooth diffeomorphisms, denoted by $\mathcal{L}_x : T_x\mathcal{F} \to \mathcal{F}(x)$, such that for each $x \in M$,*

1. *$\mathcal{L}_{f(x)}^{-1} \circ f \circ \mathcal{L}_x : T_x\mathcal{F} \to T_{f(x)}\mathcal{F}$ is a polynomial map of sub-resonance type; and*
2. *if g is a smooth automorphism of $(M,\mathcal{F})$ that commutes with f, then $\mathcal{L}_{g(x)}^{-1} \circ g \circ \mathcal{L}_x$ is also polynomial of sub-resonance type.*

Furthermore, if the $\mathcal{V}$-splitting associated to f is C^k, the diffeomorphisms $\mathcal{L}_x$ can be choosen so as to be C^k differentiable in x, as x varies along leaves.

Notice that, in the setting of a continuous lamination adopted here, the mappings $\mathcal{L}_x$ are global diffeomorphisms.

The relationship between Theorem 3 and the previous two can be seen as follows. Let $P_{\mathcal{V}}$ denote the invariant $H_{\mathcal{V}}$-reduction of $F(M,\mathcal{F})$ obtained in Theorem 1. Let $H_{\mathcal{V}}^1$ be the projection $H_{\mathcal{V}}$ into the linear group $GL(n,\mathbb{R})$, which is $GL(n,\mathbb{R}) \cap H_{\mathcal{V}}$. Form the associated bundle $Q := P_{\mathcal{V}} \times_{H_{\mathcal{V}}} H_{\mathcal{V}}/H_{\mathcal{V}}^1$. This is a (foliated) fiber bundle over M defined as the space of orbits for the product action of $H_{\mathcal{V}}$ on $P_{\mathcal{V}} \times H_{\mathcal{V}}/H_{\mathcal{V}}^1$. As it has contractible fibers, it admits continuous global sections. Let ζ be a continuous section which is C^k differentiable along leaves of $\mathcal{F}$, for some $k \geq 0$. Then for each $x \in M$ choose a representative ϕ_x of $\zeta(x)$ and define

$$\mathcal{L}_x := \phi_x \circ (d\phi_x)_0^{-1} : T_x\mathcal{F} \to \mathcal{F}(x). \tag{5}$$

If the spectral partition is trivial (in which case the narrow band condition amounts to half-pinching), then $H_{\mathcal{V}} = H_{\mathcal{V}}^1$, so that $Q = M$ and ζ is unique. Therefore, the following remark holds.

Remark 4. *When $H_{\mathcal{V}}$ is a first order group, the maps $\mathcal{L}_x$ of Theorem 3 are uniquely determined.*

We would like now to indicate how the non-stationary normal forms theorem is related to certain connection-like invariant structures. But before explaining it in general, it may be instructive to take a look at this structure in the simplest conceivable setting, namely, a contraction of $\mathbb{R}$ near a fixed point. This is, of course, a rather trivial special case, but the bare outline of the general narrative is already apparent.

Let $f : \mathbb{R} \to \mathbb{R}$ be a (local) diffeomorphism fixing 0 such that $f'(0) = \lambda \in (0,1)$. First observe that linearizing f near 0 amounts to finding an f-invariant smooth connection ∇ on the tangent bundle of $M = \mathbb{R}$, on a neighborhood of 0: the exponential map of ∇, $\exp_0 : T_0M \to M$, intertwines f and its derivative map $df_0 : T_0M \to T_0M$; that is, $f(\exp_0 u) = \exp_0 df_0 u$, due to the f-invariance of ∇. By this remark, the local normal form theorem can be restated as a theorem about the existence of an ordinary f-invariant covariant derivative, which is obtained by the following application of a well-known trick in hyperbolic dynamics. (This trick is essentially all there is to the proof of Proposition 11.) Let ∇ stand for an arbitrary smooth connection on M and denote by X the standard coordinate vector field d/dx. Write $\Gamma := \nabla_X X$, the sole Christoffel symbol of ∇. Under f, Γ transforms according to the formula $\Gamma^f = f'\Gamma \circ f + f''/f'$. Finding an invariant ∇ thus amounts to solving the equation $\Gamma = f'\Gamma \circ f + f''/f'$. By iterating this equation one obtains the unique solution as a convergent series given by $\Gamma = \varphi + f'\varphi \circ f + (f^2)'\varphi \circ f^2 + \cdots$, where $\varphi := f''/f'$. If g is a smooth (local) diffeomorphism of M commuting with f, then $g^*\nabla$, the pull-back of the f-invariant ∇ under g, is immediately seen to also be f-invariant, whence $g^*\nabla = \nabla$ as the invariant connection is unique. Consequently, $g(\exp_0 u) = \exp_0(dg_0 u)$ also holds, which is the conclusion of the centralizer theorem. Much of the content of this paper has to do with obtaining the replacement for ∇ under the general hypotheses of the sub-resonance normal forms theorem.

For the general case, it will be useful to make the following definitions.

Definition 5. *Let P be a C^k-differentiable principal H-bundle over $(M,\mathcal{F})$, $k \geq 0$, and suppose that P is an H-reduction of a principal G-bundle F. Denote by J^1F the space of 1-jets of local sections of F. This space can be regarded as a fiber bundle over F, with projection map $(j^1\sigma)_x \mapsto \sigma(x)$. Let η be a continuous section of the pull-back (restriction) of J^1F to P.*

1. *With no further restrictions, we say that η is a* quasi-connection *on P.*
2. *If η takes values in J^1P (for $k \geq 1$), we say that η is a* generalized connection *on P.*
3. *The section η is a* principal connection *on P if it is an H-invariant generalized connection.*

Notions such as parallel transport and curvature make sense for a generalized connection η. If $\gamma(t)$, $0 \leq t \leq 1$, is a piecewise differentiable curve on a leaf of $\mathcal{F}$ and $\sigma \in P$ lies on the fiber above $\gamma(0)$, then the parallel transport of σ along γ, relative to η, is the unique curve $\sigma(t)$ above $\gamma(t)$ such that the vector $\sigma'(t)$ belongs to $\eta(\sigma(t))$ for each t. Thus parallel translation is obtained by solving an ordinary differential equation on P.

The curvature of η is a two-form on P taking values in the Lie algebra of H. It is defined as follows. Given vector fields X,Y on P, let X^H, Y^H be their components along the n-plane η, and define $\Omega(X,Y) = -[X^H,Y^H]^V$, where the upper-script V denotes the vertical component of the Lie bracket. The vertical component of a vector is naturally associated to an element of the Lie algebra of H. If Ω vanishes identically, then η is an integrable distribution in P. (In particular, parallel transport along small closed curves is trivial.) We say, then, that the generalized connection is *flat*.

We now explain how these concepts are related to normal forms.

Let $F^r(M,\mathcal{F})$ denote the bundle of r-th order frames over $(M,\mathcal{F})$, obtained as the quotient of $F(M,\mathcal{F})$ by the equivalence relation that identifies diffeomorphisms with same r-jet at 0. This is a principal bundle with group G^r of r-jets at 0 of local diffeomorphisms of $\mathbb{R}^n$ fixing 0, where n is the leaf dimension of $\mathcal{F}$. Note that G^r is the quotient of $\mathcal{D}(\mathbb{R}^n)_0$ by the above r-jet equivalence relation.

Let $H^r_{\mathcal{V}}$ represent the projection of the sub-resonance group $H_{\mathcal{V}}$ into G^r. As already remarked, elements of $H_{\mathcal{V}}$ are polynomial maps of degree no greater than $d_{\mathcal{V}}$, so that $H^r_{\mathcal{V}}$ is isomorphic to $H_{\mathcal{V}}$ whenever $r \geq d_{\mathcal{V}}$. Let $P_{\mathcal{V}}$ be the invariant $H_{\mathcal{V}}$-reduction of $F(M,\mathcal{F})$ obtained in Theorem 1, and let $P^r_{\mathcal{V}}$ be the jet projection of $P_{\mathcal{V}}$ into $F^r(M,\mathcal{F})$. We thus obtain a sequence of $H^r_{\mathcal{V}}$-reductions:

$$\cdots \to P^{r+1}_{\mathcal{V}} \to P^r_{\mathcal{V}} \to \cdots \to P^1_{\mathcal{V}} \to M \tag{6}$$

such that the jet projection $\pi^{r+1}_r : P^{r+1}_{\mathcal{V}} \to P^r_{\mathcal{V}}$ is a homeomorphism for each r such that $r \geq d_{\mathcal{V}}$. Let $\eta^r : P^r_{\mathcal{V}} \to P^{r+1}_{\mathcal{V}}$ denote the inverse of this homeomorphism.

To each $\sigma \in P^r_{\mathcal{V}}$, $\eta^r(\sigma)$ can be regarded as an element of the bundle $J^1F^r(M,\mathcal{V})|_{P^r_{\mathcal{V}}}$ of 1-jets of local sections of $F^r(M,\mathcal{V})$, restricted (i.e., pulled-back) to $P^r_{\mathcal{V}}$. This will be explained in the proof of the next theorem.

Theorem 6 contains our main geometric remark.

Theorem 6. *Let $G \subset Aut(M,\mathcal{F})$ be a group whose center contains a contraction and $\mathcal{V} = \{(V_i,I_i), i = 1,\ldots,l\}$ an associated spectral partition satisfying the narrow band spectrum condition. Suppose* *that the $\mathcal{V}$-splitting of $T\mathcal{F}$ is C^k, $k \geq 0$. Fix an integer $r \geq d_{\mathcal{V}}$, and let η^r be as defined above. Then the following hold:*

(1) *η^r is a G-invariant quasi-connection on $P^r_{\mathcal{V}}$;*
(2) *if $k \geq 1$, then η^r is a G-invariant, flat, C^k generalized connection on $P^r_{\mathcal{V}}$;*
(3) *if $l = 1$ and $d_{\mathcal{V}} = 1$, then η^1 is a smooth G-invariant, flat, principal connection on $P^1_{\mathcal{V}}$.*

In all cases, η^r is C^k.

We can now see how Theorem 6 relates to Theorem1. Suppose that the $\mathcal{V}$-splitting of $T\mathcal{F}$ is C^k, $k \geq 1$, in addition to the general assumptions of Theorem 6. Therefore η^r is a flat, C^k, generalized connection on the C^k bundle $P^r_{\mathcal{V}}$. Since an isometry of the connection preserves $P^{r+1}_{\mathcal{V}}$, expressed in terms of elements this bundle the isometry assumes normal form up to order $r+1$. More precisely, given $\sigma_1, \sigma_2 \in P^{r+1}_{\mathcal{V}}$, and an isometry f, then $f^{(r+1)}(\sigma_1) = \sigma_2 h$, where $f^{(r+1)}$ is the map f induces on $P^{r+1}_{\mathcal{V}}$ and $h \in H^{r+1}_{\mathcal{V}}$ is the infinitesimal normal form of f. To obtain normal forms not only infinitesimally but on whole leaves, we proceed as follows.

By the Frobenius theorem, through each $\eta_0 \in (P^r_{\mathcal{V}})_{x_0}$ passes a unique local C^k section, $x \mapsto \eta(x)$, tangent to $\eta(\sigma(x))$ for each x near x_0. It can be shown that the section is holonomic; that is, it comes from a local diffeomorphism ϕ from a neighborhood of $0 \in \mathbb{R}^n$ into the leaf of $\mathcal{F}$ containing the base point of σ. (Denoting by $t_y : \mathbb{R}^n \to \mathbb{R}^n$ the translation in $\mathbb{R}^n$ by y, then the section σ has the form $\sigma(x) = j^r(\phi \circ t_{\phi^{-1}(x)})_0$, for x near x_0. Section 5 of [Fe3], in particular Lemma 5.4, has more details related to this.) Making use of the contraction element of G it can be shown that ϕ extends to a diffeomorphism from $\mathbb{R}^n$ to a leaf of $\mathcal{F}$. If now ϕ_1 and ϕ_2 are the diffeomorphisms associated to σ_1 and σ_2, then instead of $f^{(r+1)}(\sigma_1) = \sigma_2 h$ we have $f \circ \phi_1 = \phi_2 \circ h$.

Also note that $\psi := \phi_2 \circ \phi_1^{-1}$ is a diffeomorphism from $\mathcal{F}(x_1)$ to $\mathcal{F}(x_2)$ mapping $x_1 = \phi_1(0)$ to $x_2 = \phi_2(2)$. It is an *isometry* of the generalized connection on those leaves. In particular, the group of isometries of η acts transitively on each leaf of $\mathcal{F}$. This makes each $\mathcal{F}(x)$ a homogeneous manifold of the form $L_{\mathcal{V}}/H_{\mathcal{V}}$, where $L_{\mathcal{V}}$ is isomorphic to the group of diffeomorphisms of $\mathbb{R}^n$ generated by $H_{\mathcal{V}}$ and the translations t_x, $x \in \mathbb{R}^n$. Note that the isomorphisms depend on a choice of x. As a manifold (but not as a group), $L_{\mathcal{V}}$ is the product of $H_{\mathcal{V}}$ and $\mathbb{R}^n$. If $r = 1$, as in part 3 of the theorem, the geometry of the leaves of $\mathcal{F}$ corresponds to the (Klein) geometry of $\mathbb{R}^n$ defined by the homogeneous space $(GL(n,\mathbb{R}) \ltimes \mathbb{R}^n)/GL(n,\mathbb{R})$.

The remaining of the paper is dedicated to proving Theorem 6. This will require reworking the finite-jet part of the proof in [GK]. For this reason, the normal forms theorem up to finite jets will be proved here along the way to Theorem 6.

3. The $\mathcal{V}$-stable Group

3.1. **Generalities about Polynomial Maps.** A map $f : V \to W$ between finite dimensional vector spaces is *polynomial* if for any given basis $\{v_1, \cdots, v_n\}$ of V and $\{w_1, \cdots, w_m\}$ of W there are polynomials $f_j(x_1, \cdots, x_n)$, $j = 1, \cdots, m$, such that $f(x_1 v_1 + \cdots + x_n v_n) = \sum_j f_j(x_1, \cdots, x_n) w_j$.

We write $x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n}$ for a multi-index $\nu := (\nu_1, \cdots, \nu_n)$ in $\{0, 1, 2, \cdots\}^n$ and $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. The degree of the monomial x^ν is $|\nu| = \nu_1 + \cdots + \nu_n$. A homogeneous polynomial of degree r is a linear combination of monomials of degree r. A polynomial map uniquely decomposes as a sum of homogeneous components.

The *first polarization*, $P_1 f : V \times V \to W$, of a homogeneous polynomial map $f : V \to W$ is defined by

$$P_1 f(u, v) := \frac{d}{dt} f(u + tv)_{t=0}. \tag{7}$$

Is is not difficult to show that if f has degree r then $P_1 f(v,v) = r f(v)$. The i-th polarization of f, $P_i f : V^i \to W$, is obtained by taking the first polarization of $u \mapsto P_{i-1} f(u, v_1, \cdots, v_{i-2})$. If r is the degree of f, we call the symmetric r-linear map $P_r f$ the *complete polarization* of f.

It is often convenient to regard $P_r f$ as an element of the tensor space $S_r(V^*) \otimes W$, where $S_r(V^*)$ denotes the subspace of symmetric elements of the r-th order tensor power of the dual space V^*. It can be shown that $r! f(v) = P_r f(v, \cdots, v)$ for f homogeneous of degree r.

Suppose that V decomposes as a direct sum: $V = V_1 \oplus \cdots \oplus V_l$. Let $F : V^r \to V$ be an r-linear map and denote by $\pi_j : V \to V_j$ the natural projection. Define $\Pi^j_{i_1 \cdots i_r} F$ by

$$\Pi^j_{i_1 \cdots i_r} F(v_1, \cdots, v_r) := \pi_j F(\pi_{i_1} v_1, \cdots, \pi_{i_r} v_r), \tag{8}$$

where $v_1, \cdots, v_r \in V$. This yields the decomposition $F = \sum_{j, i_1, \cdots, i_r = 1}^{l} \Pi^j_{i_1 \cdots i_r} F$.

3.2. **The Group $H_{\mathcal{V}}$.** We fix now a spectral partition $\mathcal{V}$ for $V = \mathbb{R}^n$. A sequence of linear isomorphisms, $L_j : V \to V$, as well as the associated sequence of iterations $L(j) = L_j \cdots L_1$, $j = 1, 2, \cdots$, will be called *$\mathcal{V}$-adapted* if: (i) L_j respects the direct sum decomposition for all j, and (ii) for each $i = 1, \cdots, l$ and all non-zero $v \in V_i$, the inequalities

$$a_i \le \liminf_{n\to\infty} \frac{1}{n} ln\|L(n)v\| \le \limsup_{n\to\infty} \frac{1}{n} ln\|L(n)v\| \le b_l \tag{9}$$

hold. Clearly, being $\mathcal{V}$-adapted does not depend on the choice of norm, $\|\cdot\|$, on V.

We wish to consider maps from $V^k = V \times \cdots \times V$ into V which do not grow under the action of a $\mathcal{V}$-adapted sequence. Such maps will be called *$\mathcal{V}$-stable*. More precisely, consider the action of a linear isomorphism $A : V \to V$ on an arbitrary map $f : V^k \to V$ defined by $A_* f : V^k \to V$ such that

$$(A_* f)(v_1, \cdots, v_k) := A f(A^{-1} v_1, \cdots, A^{-1} v_k). \tag{10}$$

Then f is $\mathcal{V}$-stable if, by definition, there exists a $\mathcal{V}$-adapted sequence $L(n)$ and, for each compact subset $K \subset V^k$ we can find $C > 0$ such that $\|(L(n)_* f)(v_1, \cdots, v_k)\| \le C$, for all $(v_1, \cdots, v_k) \in K$ and $n = 1, 2, \cdots$. The term *L-stable* will also be used when explicit reference to the adapted sequence is required.

Lemma 7. *Let $\mathcal{V} = \{(V_i, I_i) : i = 1, \cdots, l\}$ be a spectral partition of V that satisfies the narrow band condition. Then, the following hold:*

(1) *if $a_j \le b_{i_1} + \cdots + b_{i_k}$ then $\lambda_j \le \lambda_{i_1} + \cdots + \lambda_{i_k}$, for any given set of indices $j, i_1, \cdots, i_k$;*
(2) *any $\mathcal{V}$-stable homogeneous polynomial map is also $\{T^n_{\mathcal{V}}\}$-stable;*
(3) *the degree of a $\mathcal{V}$-stable homogeneous polynomial is bounded above by a_1/b_l;*
(4) *the degree of a $\{T^n_{\mathcal{V}}\}$-stable homogeneous polynomial is bounded above by $d_{\mathcal{V}}$;*
(5) *if f is $\mathcal{V}$-stable, and F is the homogeneous component of f of degree k, then*

$$b_{i_1} + \cdots + b_{i_k} < a_j \Rightarrow \Pi^j_{i_1 \cdots i_k} F = 0;$$

(6) *if f is $\{T^n_{\mathcal{V}}\}$-stable, and F is the homogeneous component of f of degree k, then*

$$\lambda_{i_1} + \cdots + \lambda_{i_k} < \lambda_j \Rightarrow \Pi^j_{i_1 \cdots i_k} F = 0.$$

Proof. Let h be a $\mathcal{V}$-stable homogeneous polynomial map of degree k and H its complete polarization. It suffices for proving 2 to check that each component $\Pi^j_{i_1 \cdots i_k} H$ is $\{T^n_{\mathcal{V}}\}$-stable. Thus suppose, without losing generality, $H = \Pi^j_{i_1 \cdots i_k} H$ and that it is non-zero. By definition, H is $\{L(n)\}$-stable, where $L(n), n = 1, 2, \cdots$, is a $\mathcal{V}$-adapted sequence of isomorphisms of V. Choose v_s in V_{i_s}, for each $s = 1, \cdots, k$, so that $H(v_1, \cdots, v_k)$ is a non-zero vector (necessarily in V_j). Write $w_i(n) = v_i/\|L(n)v_i\|$.

The definition of $L(n)_*H$ implies $L(n)v' = \|L(n)v_1\|\cdots\|L(n)v_k\|u_n$, where $v' = H(v_1,\cdots,v_k)$ and $u_n = (L(n)_*H)(w_1(n),\cdots,w_k(n))$. But $\|u_n\| \le C$ for some constant C independent of n. Therefore $\|L(n)v'\| \le C\|L(n)v_1\|\cdots\|L(n)v_k\|$. Taking logarithms, dividing by n, and passing to the lim sup and lim inf as in 9, results in $a_j \le b_{i_1} + \cdots + b_{i_k}$, which proves part 5.

It is convenient to rewrite the last inequality as

$$a_j \le m_1 b_1 + \cdots + m_l b_l, \tag{11}$$

where m_s is the number of times (possibly 0) that b_s appears on the right-hand side of the first inequality. Since

$$T^n_{\mathcal{V}*}H = \exp\left(n\lambda_j - n\sum_{i=1}^{l} m_i\lambda_i\right)H, \tag{12}$$

part 2 of the lemma is a consequence of part 1. That is, we need to show that

$$\lambda_j - \sum_{i=1}^{l} m_i\lambda_i \le 0. \tag{13}$$

Notice that, in 11, $m_i = 0$ for $i < j$, and by the narrow band condition ($a_j > b_j + b_l$) inequality 11 reduces to $a_j \le b_j$ whenever $m_j \neq 0$. Consequently, 13 holds if $m_j \neq 0$, since in this case $m_j = 1$ and $m_i = 0$ for $i \neq j$. Therefore, we may assume that $m_1 = \cdots = m_j = 0$. Then

$$\begin{aligned}
\lambda_j &= \frac{a_j}{b_{j+1}}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{l-1}}{b_l}a_l \\
&\le \sum_{i=j+1}^{l} m_i\frac{b_i}{b_{j+1}}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{l-1}}{b_l}a_l \\
&= m_{j+1}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{l-1}}{b_l}a_l + \sum_{i=j+2}^{l} m_i\left(\frac{b_i}{b_{j+1}}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{i-1}}{b_i}\right)\frac{a_i}{b_{i+1}}\cdots\frac{a_{l-1}}{b_l}a_l \\
&= m_{j+1}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{l-1}}{b_l}a_l + \sum_{i=j+2}^{l} m_i\left(\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{i-1}}{b_{i-1}}\right)\frac{a_i}{b_{i+1}}\cdots\frac{a_{l-1}}{b_l}a_l \\
&\le m_{j+1}\frac{a_{j+1}}{b_{j+2}}\cdots\frac{a_{l-1}}{b_l}a_l + \sum_{i=j+2}^{l} m_i\frac{a_i}{b_{i+1}}\cdots\frac{a_{l-1}}{b_l}a_l \\
&\le \sum_{i=j+1}^{l} m_i\lambda_i,
\end{aligned}$$

which is what is needed to check that $\|T^n_*H\|$ is bounded.

To see part 3, notice that, as H is non-zero only if $a_j \le b_{i_1} + \cdots + b_{i_k}$, we have $a_1 \le kb_l$. As $b_l < 0$, this shows that $k \le a_1/b_l$, proving that a_1/b_l is an upper-bound for the degree. Part 4 holds for a similar reason. □

Let $\mathrm{Pol}_{\mathcal{V}}$ denote the set of all $\{T^n_{\mathcal{V}}\}$-stable polynomial maps from V to itself fixing 0. Since $T_{\mathcal{V}*}(f\circ g) = T_{\mathcal{V}*}f\circ T_{\mathcal{V}*}g$, it is clear that $\mathrm{Pol}_{\mathcal{V}}$ is closed under composition of maps. Furthermore, $f \in \mathrm{Pol}_{\mathcal{V}}$ iff each f_i in the decomposition $f = f_1 + f_2 + \cdots + f_r$ into homogeneous terms also lies in $\mathrm{Pol}_{\mathcal{V}}$. This can be seen as follows. If f is $\mathcal{V}$-stable, then f_c, defined by $f_c(v) := f(cv)$, is also $\mathcal{V}$-stable map for each c. But $f_c = f_0 + cf_1 + \cdots + c^r f^r$. Taking $c_0 = 1, c_2, \cdots, c_r$ all distinct, it is possible to write each f_i as a linear combination of the f_{c_i}. Therefore, each f_i is $\mathcal{V}$-stable if so is f.

Since stable maps have degree bounded by $d_{\mathcal{V}}$, the space $\mathrm{Pol}_{\mathcal{V}}$ is finite dimensional and, as just remarked, closed under composition. Now define

$$H_{\mathcal{V}} = \{f \in \mathrm{Pol}_{\mathcal{V}} \mid f_1 \text{ is non-singular}\}. \tag{14}$$

It will be shown shortly that $H_{\mathcal{V}}$ is a Lie group, which we call the *$\mathcal{V}$-stable group*.

The following remarks and notations will be used in verifying this claim. First observe that the subgroup $GL(\mathcal{V})$ of $GL(V)$ stabilizing the flag $W_i = V_i \oplus V_{i-1} \oplus \cdots \oplus V_1$, $i = 1, \cdots, l$, is contained in $H_{\mathcal{V}}$. This is, in fact, the group generated by the elements f_1, for $f \in H_{\mathcal{V}}$. Let $\mathrm{Pol}_{\mathcal{V}}(i)$ be the subspace of $\mathrm{Pol}_{\mathcal{V}}$ spanned by all homogeneous polynomial maps of degree greater than or equal to i. Then

$$\mathrm{Pol}_{\mathcal{V}}(i) \circ \mathrm{Pol}_{\mathcal{V}}(j) \subset \mathrm{Pol}_{\mathcal{V}}(ij), \tag{15}$$

where the left-hand side indicates the set of $f \circ g$, for $f \in \mathrm{Pol}_{\mathcal{V}}(i)$ and $g \in \mathrm{Pol}_{\mathcal{V}}(j)$. We denote by $I + \mathrm{Pol}_{\mathcal{V}}(i)$ the set of $f \in P_{\mathcal{V}}$ of the form $f(x) = x + h(x)$, $h \in \mathrm{Pol}_{\mathcal{V}}(i)$.

Proposition 8. *$H_{\mathcal{V}}$, with the operations of composition and inverse of maps, is a group.*

Proof. Given $f = f_1 + f_2 + \cdots + f_k \in H_{\mathcal{V}}$, all that is needed is to find f^{-1} in $H_{\mathcal{V}}$. The proof uses an inductive argument that obtains in the end the inverse map as a product $f^{-1} = h_m \circ h_{m-1} \circ \cdots \circ h_1$, $h_j \in I + \mathrm{Pol}_{\mathcal{V}}(j)$. Set $h_1 = f_1^{-1}$. As $f_1 \in GL(\mathcal{V})$, $h_1 \circ f \in I + \mathrm{Pol}_{\mathcal{V}}(2)$. Assume that we have found $h_i \in I + \mathrm{Pol}_{\mathcal{V}}(i)$, $2 \le i \le j-1$, such that $g \circ f \in I + \mathrm{Pol}_{\mathcal{V}}(j)$, where $g = h_{j-1} \circ \cdots \circ h_1$. Write $(g \circ f)(x) = x + u(x)$, for some $u \in \mathrm{Pol}_{\mathcal{V}}(j)$. Then, it is easily checked that $(I - u) \circ g \circ f$ lies in $I + \mathrm{Pol}_{\mathcal{V}}(j+1)$, and we choose $h_j = I - u$. For m big enough we finally reach $\mathrm{Pol}_{\mathcal{V}}(m) = 0$, so that $h_m \circ \cdots \circ h_1 \circ f = I$. □

Observe that if $l = 1$, the narrow band condition reads $-a < -2b$ (half-pinching). In this case, no f in $H_{\mathcal{V}}$ can have degree greater than 1, since the half-pinching condition precludes non-trivial relations of the type 11. Also notice that under half-pinching the map T of Lemma 7 is scalar, so $H_{\mathcal{V}}$ is the full group $GL(V)$.

As another example, suppose that $V = V_1 \oplus V_2$ and the integer part of a_1/b_2 is 2, so that polynomials in $H_{\mathcal{V}}$ have degree at most 2. Then any $f \in H_{\mathcal{V}}$ must have the form

$$f(v) = (L_1 v + Q(v_2, v_2), L_2 v_2), \tag{16}$$

(we are using here notation that is explained at the beginning of the next section) where L_1 maps V to V_1, L_2 maps V_2 to V_2, and Q maps $V_2 \times V_2$ to V_1. Notice that maps of this form, for $L = (L_1, L_2)$ non-singular, indeed form a group of diffeomorphisms of V. It can be regarded as a subgroup of the jet group G^r (defined below) for any $r \ge 2$.

It is interesting to note that the normalizer group of $H_{\mathcal{V}}$ in G^r, for each $r \ge d_{\mathcal{V}}$, is $H_{\mathcal{V}}$ itself. This is a consequence of Proposition 11 given later.

We have defined $H_{\mathcal{V}}$ as a group of diffeomorphisms fixing the origin of $\mathbb{R}^n$. By dropping the latter condition we obtain the *extended $\mathcal{V}$-stable group*, $L_{\mathcal{V}}$, which is generated by $H_{\mathcal{V}}$ and the translations in $\mathbb{R}^n$. As manifolds (but not as groups), $L_{\mathcal{V}} = H_{\mathcal{V}} \times \mathbb{R}^n$, so that $L_{\mathcal{V}}/H_{\mathcal{V}}$ is homeomorphic to $\mathbb{R}^n$.

Before proceeding with the description of the invariant $H_{\mathcal{V}}$-structures claimed in the normal forms theorem, we need to review some basic facts concerning jet spaces and jet groups.

3.3. **Generalities about Jet Groups.** Let L and N be smooth manifolds and f, g smooth maps from some neighborhood of a point $x \in L$ into N. We say that f and g represent the same r-jet at x if $f(x) = g(x) = y$ and, with respect to some choice of smooth coordinates about x and y, all partial derivatives at x of f and g up to order r agree. More precisely, let t^i, $1 \le i \le n = \dim L$, be smooth coordinates near x, let u^j, $1 \le j \le \dim N$, be smooth coordinates near y, and represent by D_i the partial derivative with respect to t_i. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be a vector of nonnegative integers and define $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$. Then f and g represent the same r-jet at x if for each i and α such that $|\alpha| := \alpha_1 + \cdots + \alpha_n \le r$

$$D^\alpha(u^i \circ f)_x = D^\alpha(u^i \circ g)_x. \tag{17}$$

This defines an equivalence relation on C^r local maps that does not depend on the choice of coordinates. The equivalence class represented by f is called the *r-jet* of f at x. It will be denoted by $j^r f_x$. The r-jets of local C^r maps comprise the r-jet space $J^r(L,N)$. This is a smooth manifold. Smooth local coordinates, u^i_α, can be set on $J^r(L,N)$ by defining $u^i_\alpha(j^r f_x) := D^\alpha(u^i \circ f)_x$. With this choice of coordinates one shows that $J^r(L,N)$ is a locally trivial fiber bundle over L. The fiber above $x \in L$ will be denoted by $J^r(L,N)_x$. We can also regard $J(L,N)$ as a fiber bundle over $L \times N$. In this case, the fiber $J^r(L,N)_{xy}$ consists of all $j^r f_x$ for which $f(x) = y$. Whenever $s \ge r$, there is a natural surjection $\pi^s_r : J^s(L,N) \to J^r(L,N)$, mapping $j^s f_x$ to $j^r f_x$. This will be referred to as the *jet projection*. The same notation, π^s_r, will be used irrespective of the particular L,N at hand.

We now turn to jet groups. The collection of all r-jets at 0 of local diffeomorphisms of a finite dimensional real linear space V fixing 0 constitutes a Lie group, denoted here $G^r(V)$, or simply G^r. Notice that G^1 is the general linear group $GL(V)$.

Taking G^2 as an example, note that the second order jet of a local diffeomorphism $\varphi : V \to V$ fixing the origin of V is completely specified by a pair (A,α) such that $A \in GL(V)$ and $\alpha \in N^2_1 := S_2(V^*) \otimes V$ is a V-valued symmetric bilinear form. Indeed, we can think of (A,α) as the Taylor polynomial of φ of order 2. (More precisely, the second degree V-valued polynomial approximating φ is $Ax + \alpha(x,x)$.) If φ and ψ have Taylor polynomials (A,α) and (B,β), respectively, then the composition $\varphi \circ \psi$ has Taylor polynomial $(AB, \alpha(B\cdot, B\cdot) + \beta)$. The group G^2, therefore, can be given the following description. $G^2 = GL(V) \ltimes N^2_1$, with multiplication

$$(A,\alpha)(B,\beta) := (AB, \alpha(B\cdot, B\cdot) + A\beta). \tag{18}$$

It follows that the inverse operation is

$$(A,\alpha)^{-1} := (A^{-1}, -A^{-1}\alpha(A^{-1}\cdot, A^{-1}\cdot)). \tag{19}$$

It will be necessary to describe in some detail the general structure of the groups $G^r(V)$. First note that the jet projections $\pi^s_r : G^s \to G^r$ form a tower of homomorphisms:

$$\cdots \to G^{s+1} \to G^s \to \cdots \to G^1 \to \{e\}. \tag{20}$$

The normal subgroups $N^s_l = \ker \pi^s_l$ form a sequence

$$N^s_s = \{e\} \subset N^s_{s-1} \subset \cdots \subset N^s_1 \subset N^s_0 = G^s. \tag{21}$$

The group G^s is an open subset of the Euclidean space $J^s(V,V)_{00}$. Indeed an $f \in G^s(V)$ is a polynomial map $f = f_1 + f_2 + \cdots + f_s$ having invertible linear term f_1. The homogeneous term of i-th degree, f_i, can be regarded (by complete polarization) as an element of the vector space $S_i(V,V) := S_i(V^*) \otimes V$ of symmetric i-linear maps from V to V. As a vector space, N^s_r is the direct sum $S_{r+1}(V,V) \oplus \cdots \oplus S_s(V,V)$, and the group operation is given by composition of maps followed by truncation to degree s.

Elements of G^r are written as r-tuples $\alpha = (\alpha_1, \cdots, \alpha_r)$, with $\alpha_i \in S_i(V,V)$. Given $\alpha_i \in S_i(V,V)$ and $\beta_j \in S_j(V,V)$, the notation $\alpha_i\beta_j$ refers to the element of $S_{ij}(V,V)$ obtained by complete polarization of

$$f(x) = \alpha_i(\beta_j(x, \cdots, x), \cdots, \beta_j(x, \cdots, x)). \tag{22}$$

Mainly needed are the compositions $\alpha_1\beta_r$ and $\alpha_r\beta_1$. These are simply

$$(\alpha_r\beta_1)(v_1,\cdots,v_r) = \alpha_r(\beta_1 v_1,\cdots,\beta_1 v_r) \tag{23}$$

$$(\alpha_1\beta_r)(v_1,\cdots,v_r) = \alpha_1\beta_r(v_1,\cdots,v_r). \tag{24}$$

The multiplication in G^r will be written simply as $\alpha\beta$. (This involves composition and truncation of terms of order greater than r.)

The following simple identities are useful:

$$(\alpha_1,\cdots,\alpha_r,\alpha_{r+1}) = (\alpha_1,\cdots,\alpha_r,0)(I,0,\cdots,0,\alpha_1^{-1}\alpha_{r+1}) \tag{25}$$

$$= (I,0,\cdots,0,\alpha_{r+1}\alpha_1^{-1})(\alpha_1,\cdots,\alpha_r,0). \tag{26}$$

It is now easy to show that for any $\alpha = (A,\alpha_2,\cdots,\alpha_r,\Phi)$, $\beta = (I,0,\cdots,0,\Gamma)$, $\gamma = (A,\alpha_2,\cdots,\alpha_r,\Psi)$, all in G^{r+1}, we have

$$\alpha\beta\gamma^{-1} = (I,0,\cdots,0,A\Gamma A^{-1} + (\Phi-\Psi)A^{-1}). \tag{27}$$

Let H be a closed subgroup of G^{r+1}. Define $H^s := \pi_s^{r+1}(H)$ for $s \leq r$. Write $H' := (\pi_r^{r+1})^{-1}(H^r)$, a closed subgroup of G^{r+1} containing H. We wish to consider the action of H' on the quotient $H\backslash H'$. For this purpose, define $N := \ker(\pi_r^{r+1}|_H)$ and $N' := \ker(\pi_r^{r+1}|_{H'})$. (Observe that $N' = N_r^{r+1}$.) Choose a smooth section $\xi : H^r \to H$ of the N-bundle H over H^r. (Such a section exists since the fiber, N, is a contractible space.) We write $\xi(h) = (h,\xi_{r+1}(h))$. Let $\eta : H' \to N'$ be the smooth map defined by

$$\eta(h') := \xi(\pi_r^{r+1}(h'))^{-1}h' = (I,0,\cdots,0,h_1^{-1}h'_{r+1} - h_1^{-1}\xi_{r+1}(h)), \tag{28}$$

for $h' \in H'$ and $h = \pi_r^{r+1}(h')$. Notice that $\eta(hh')\eta(h')^{-1} \in N$, for $h' \in H'$ and $h \in H$, so that the coset $N\eta(h') \in N\backslash N'$ only depends on the class of h' in $H\backslash H'$. Writing $\eta = (I,0,\cdots,0,\eta_{r+1})$, the definition of η gives

$$\eta_{r+1}(h') = A^{-1}(\Phi - \xi_{r+1}(h)), \tag{29}$$

for $h' = (A,h_2,\cdots,h_r,\Phi)$.

The group N(resp., N') will be identified with the linear subspace W (resp., W') of $S_{r+1}(V,V)$ consisting of Γ such that $(I,0,\cdots,0,\Gamma)$ is in N (resp., N'), and $N\backslash N'$ with the quotient space $W\backslash W'$. (Notice that $W' = S_{r+1}(V,V)$.) It is easily checked that

$$\eta_{r+1}(g'h') + W = A^{-1}\eta_{r+1}(g')A + \eta_{r+1}(h') + W, \tag{30}$$

for $g',h' \in H'$, $g = \pi_r^{r+1}(g'), h = \pi_r^{r+1}(h')$, and $A = \pi_1^{r+1}(h')$.

The following proposition is an elementary consequence of these various facts and definitions.

Proposition 9. *The map $\Xi : Hh' \in H\backslash H' \mapsto [\eta_{r+1}(h')] \in W\backslash W'$ is a smooth diffeomorphism, with inverse $Nn' \mapsto Hn'$. The action of H' on $H\backslash H'$ by right translations is conjugate under Ξ to the action of H' on $W\backslash W'$ given by*

$$([\Gamma],h') \mapsto [A^{-1}\Gamma A + \eta_{r+1}(h')], \tag{31}$$

for $h' \in H'$, $A = \pi_1^{r+1}(h')$ and $[\Gamma] \in W\backslash W'$.

4. GEOMETRIC STRUCTURES ON CONTINUOUS LAMINATIONS

4.1. **Continuous Laminations.** Let M be a compact metric space, $\{U_i : i \in A\}$ an open cover of M, and $h_i : U_i \to B \times T_i$ a homeomorphism for each $i \in A$, where $B = (-1,1)^n \subset \mathbb{R}^n$ and T_i is a topological space. These open sets and homeomorphisms are said to constitute an atlas for a lamination of M by n-dimensional smooth manifolds if the coordinate changes $h_{ij} = h_j \circ h_i^{-1}$, from $h_i(U_i \cap U_j)$ to $h_j(U_i \cap U_j)$ have the form

$$h_{ij}(x,t) = (\phi_{ij}(x,t), \eta_{ij}(t)), \tag{32}$$

where $\phi_{ij}(x,t)$ is smooth in x and all partial derivatives in x are continuous in t. Two atlas are equivalent if their union is also an atlas. A *lamination by smooth manifolds* is then defined as an equivalence class of atlas.

Sets of the form $h_i^{-1}(B \times \{t\})$ are called *plaques* of the lamination. A *leaf* of the lamination is a connected subset of M which is minimal for the property that whenever the set intersects a plaque, the entire plaque is contained in it. The leaf through $x \in M$ will be denoted $\mathcal{F}(x)$. The lamination itself will be denoted simply by $\mathcal{F}$.

We say that the lamination is n-dimensional if its leaves are n-dimensional. By a differentiable (C^1, C^k, smooth, etc.) function on $\mathcal{F}$ it is understood a continuous function on M whose restriction to each leaf is differentiable.

It is easy to guess what one should mean by the tangent bundle, $T\mathcal{F}$, of $\mathcal{F}$. Notice that the fiber of $T\mathcal{F}$ at x, denoted $T_x\mathcal{F}$, is the tangent space to $\mathcal{F}(x)$ at x. A *vector field* on $\mathcal{F}$ is by definition a continuous section of $T\mathcal{F}$ that restricts to a smooth vector field on each leaf. It is also not difficult to guess how the notion of a Riemannian metric on $T\mathcal{F}$ should be defined.

Fiber bundles that are attached in a natural way to smooth manifolds, such as tensor bundles, frame bundles, etc., can be defined in the lamination setting. The same is true for the various notions of geometric structure on a smooth manifold. Some of these concepts are briefly recalled later in the paper. A more detailed discussion, in the ordinary setting of smooth manifolds, is given in [Fe3]. Everything that is said there can be adapted to the present setting without difficulty.

4.2. **Frame Bundles over $\mathcal{F}$.** Geometric structures of interest in differential geometry, in particular those that will arise in this paper, are often so called H-structures of order r, which are reductions of the r-th order frame bundle of a differentiable manifold to a subbundle with structure group H. The main definitions are explained below.

Let L be an n-dimensional smooth manifold (taken shortly to be a leaf of $\mathcal{F}$). In what follows we write $V := \mathbb{R}^n$. A *frame of order r* at $x \in L$ is the r-jet at x of a smooth parametrization of L around x. A frame of order 1 at x, in particular, is naturally identified with a linear isomorphism from V onto the tangent space T_xL. In the general case, the equivalence class represented by a parametrization φ will be denoted $(j^r\varphi)_0$, the r-th jet of φ at 0.

The collection of all frames over points of L forms in a natural way a smooth manifold, called the r-th order *frame bundle of L* and denoted $F^r(L)$. This is a locally trivial fiber bundle over L and the bundle map $\pi : F^r(L) \to L$ is the base point projection $(j^r\varphi)_0 \mapsto \varphi(0)$.

Having fixed a frame $\xi = (j^r\varphi)_0$ at x, any other frame of order r at the same point is given by ξg, for some $g = (j^r f)_0 \in G^r$. By definition, $\xi g := j^r(\varphi \circ f)_0$. The map $F^r(L) \times G^r \to F^r(L)$ defined by $(\xi, g) \mapsto \xi g$ is a smooth group action that sends each fiber of $F^r(L)$ onto itself. It is clear, furthermore, that the action is transitive on each fiber. With this action $F^r(L)$ becomes a principal bundle. A smooth parametrization of an open subset $U \subset L$ can be used to trivialize $F^r(L)$ above U, making $\pi^{-1}(U) \subset F^r(L)$ isomorphic to the trivial bundle $U \times G^r$. The fiber of $F^r(L)$ above x will be denoted $F^r(L)_x$.

Let now $(M,\mathcal{F})$ be a lamination by smooth manifolds. For each $x \in M$, let $L = \mathcal{F}(x)$ be the leaf through x and write $F^r(M,\mathcal{F})_x := F^r(L)_x$. Let $F^r(M,\mathcal{F})$ be the disjoint union of the $F^r(M,\mathcal{F})_x$, for $x \in M$, and $\pi : F^r(M,\mathcal{F}) \to M$ the base point map. Then $F^r(M,\mathcal{F})$ is in a natural way a smooth lamination for which π is a smooth lamination morphism, that is, a continuous map sending leaves to leaves whose restriction to each leaf of the source lamination is smooth.

To describe an atlas for $F^r(M,\mathcal{F})$, start with a lamination atlas for $(M,\mathcal{F})$ with local maps $h_i : U_i \to B \times T_i$. Write $U_i^{(r)} = \pi^{-1}(U_i)$ and define $h_j^{(r)} : U_i^{(r)} \to F^r(B) \times T_i$ so that $h_j^{(r)}(\xi) = j^r(h \circ \phi)_0$, for each $\xi = j^r(\phi)_0 \in F^r(\mathcal{F}(x))_x$. It is a simple exercise to verify that these $\left(U_i^{(r)}, h_i^{(r)}\right)$ indeed define a lamination atlas with the desired properties.

4.3. **Geometric Structures on $\mathcal{F}$.** Let $\mathcal{K}$ be a manifold equipped with a smooth (right) action of G^r. A *geometric structure on $(M,\mathcal{F})$ of order r and type $\mathcal{K}$* is a map $\mathcal{G} : F^r(M,\mathcal{F}) \to \mathcal{K}$ that satisfies the G^r-equivariance property: $\mathcal{G}(\xi g) = \mathcal{G}(\xi)g$ for all $\xi \in F^r(M,\mathcal{F})$ and $g \in G^r$. Alternatively, it can be defined as a section of the associated bundle $F^r(M,\mathcal{F}) \times_{G_r} \mathcal{K}$. The latter is defined as the space of orbits for the action of G^r on $F^r(M,\mathcal{F}) \times \mathcal{K}$ given by $(\xi,\kappa)g := (\xi g, \kappa g)$. We say that $\mathcal{G}$ is an H-structure of order r. It will be said to be smooth (resp. C^r, real analytic, continuous, measurable, etc.) if the equivariant map $\mathcal{G}$ is smooth (resp. C^r, real analytic, continuous, measurable). Recall that any differentiability requirement is always understood to hold only along leaves.

The geometric structures of interest here are *H-structures*. An H-structure, where H is a closed subgroup of G^r, is simply an H-reduction $P \subset F^r(M,\mathcal{F})$, that is, a principal H-bundle relative to the restriction to P of the projection map of $F^r(M,\mathcal{F})$ and H-action defined by restriction of the G^r-action. To view it as a section of an associated bundle or as a G^r-equivariant map, we can take $\mathcal{K} := H\backslash G^r$ and G^r-action on $\mathcal{K}$ by right translation. Conversely, if $\mathcal{G} : F^r(M,\mathcal{F}) \to H\backslash G^r$ is an equivariant map, it is easily seen to produce an H-reduction of $F^r(M,\mathcal{F})$. In fact, $P = \mathcal{G}^{-1}([e])$, where $[e]$ represents the identity coset in $H\backslash G^r$.

As another example of geometric structure, consider linear connections. Let $\mathcal{K}$ be the space of all V-valued bilinear maps $\Gamma : V \times V \to V$, which we regard as the space of Christoffel symbols. To describe the action of G^2 on $\mathcal{K}$ first observe that G^2 is isomorphic to $GL(V) \ltimes \mathcal{K}$. Then $\mathcal{K}$ can be identified with the coset space

$$\mathcal{W} = GL(V)\backslash(GL(V) \ltimes \mathcal{K}) \tag{33}$$

by setting $\Gamma \mapsto GL(V)(I,\Gamma)$. The group G^2 naturally acts on $\mathcal{W}$ by right-multiplication, yielding the following action on $\mathcal{K}$:

$$\Gamma \cdot (A,\alpha) := A^{-1}\Gamma A + A^{-1}\alpha. \tag{34}$$

This is the familiar law of transformation of Christoffel symbols. A linear connection corresponds in fact to a G^2-equivariant map $\mathcal{G} : F^2(M,\mathcal{F}) \to \mathcal{K}$. The relationship between $\mathcal{G}$ and the notion of a covariant derivative is seen as follows. Let $\xi \in F^2(M,\mathcal{F})$, the 2-jet (at 0) of a smooth parametrization φ around $x \in L$ (with $\varphi(0) = x$ and L the leaf through x) and X_i the coordinate vector fields associated to φ. Then $(\nabla_{X_i} X_j)_x = \sum_{k=1}^n \Gamma_{ij}^k X_k$ defines a covariant derivative, where Γ_{ij}^k is the k-th component of $\mathcal{G}(\xi)(e_i,e_j)$ with respect to the standard basis $\{e_1,\cdots,e_n\}$ of V. Also notice that a symmetric (torsion-free) connection is one for which the map $\mathcal{G}$ takes values into $G^2/GL(V) \subset \mathcal{K}$.

4.4. **Affine Bundles.** Let $H \subset G^{r+1}(V)$ be a subgroup, $H^r \subset G^r(V)$ the image of H under the homomorphism $\pi_r^{r+1} : G^{r+1}(V) \to G^r(V)$, and $H' \subset G^{r+1}(V)$ the pre-image of H^r under π_r^{r+1}. Clearly $H \subset H'$. Form the homogeneous space $\mathcal{K} = H\backslash H'$ and consider the action of H' on $\mathcal{K}$ by right-translations.

The notations used in Proposition 9 will be in effect now. According to that proposition, $H\backslash H'$ is diffeomorphic to $\overline{W} := W\backslash W'$ via the map Ξ defined there, where $W' = S_{r+1}(V^*)\otimes V$ and $W \subset W'$ is an invariant subspace under conjugation by elements in H^1; furthermore, the corresponding action of H' on $\overline{W}$ is affine, given by: $([\Gamma], h') = [A^{-1}\Gamma A + \eta_{r+1}(h')]$, where $A = \pi_1^{r+1}(h')$. Recall that η_{r+1} only depends on the class represented by h' in $H\backslash H'$.

Starting with an H^r-reduction, P^r, of $F^r(M,\mathcal{F})$ define an H'-reduction, Q, of $F^{r+1}(M,\mathcal{F})$ by

$$Q = \{\xi \in F^{r+1}(M,\mathcal{F}) \mid \pi_r^{r+1}(\xi) \in P^r\}. \tag{35}$$

We wish to consider in some detail the H-reductions of Q, which are in one-to-one correspondence with sections of a certain affine bundle over M. Before introducing this affine bundle, we describe the vector bundle on which it is modeled.

Define a vector bundle $\overline{E}^{r+1}$ over M as the quotient

$$\overline{E}^{r+1} := E^{r+1}\backslash S_{r+1}(T^*\mathcal{F})\otimes T\mathcal{F}, \tag{36}$$

where E^{r+1} is the subbundle of $S_{r+1}(T^*\mathcal{F})\otimes T\mathcal{F}$ constructed as follows. Let ρ be the representation of H' in $GL(W)$ given by $\rho(h')w := AwA^{-1}$, $A = \pi_1^{r+1}(h')$ and write $E^{r+1} := (Q\times W)/{\sim_\rho}$, which is the space of orbits of the H'-action, $(\xi, w)\cdot h' := (\xi h', \rho(h')^{-1}w)$, on $Q\times H'$. (Of course, this could just as well be defined as an associated bundle to $\pi_1^{r+1}(Q) \subset F^1(M,\mathcal{F})$, rather than to Q, for the standard representation of $\pi_1^{r+1}(H') \subset GL(V)$ on the tensor space W of V; but it will be convenient to defined it this way.) Notice that E^{r+1} is indeed a subbundle of the vector bundle $S_{r+1}(T^*\mathcal{F})\otimes T\mathcal{F}$ over M, since the latter is similarly obtained from the linear action on W' (instead of W) derived from ρ. An explicit isomorphism from $(Q\times W')/\sim_\rho$ to $S_{r+1}(T^*\mathcal{F})\otimes T\mathcal{F}$ is given by: $[\xi, w] \mapsto (d\phi_0)_* w$, where $\xi = (j^{r+1}\phi)_0$ and $(d\phi_0)_*$ denotes push-forward of a tensor on V to a tensor on $T_{\phi(0)}\mathcal{F}$ via the linear map $d\phi_0$.

We now set

$$\mathrm{Aff}(\overline{E}^{r+1}) = (Q\times\overline{W})/H' \tag{37}$$

where, this time, H' acts on the product $Q\times\overline{W}$ by $(\xi, [\Gamma])\cdot h' = (\xi h', [\Gamma]h')$. (Here $[\Gamma]h'$ denotes the action on $\overline{W}$ isomorphic under Ξ to the action by right-translations on $H\backslash H'$.) The next proposition is now immediate.

Proposition 10. *Aff($\overline{E}^{r+1}$) is an affine bundle over M modeled on the vector bundle $\overline{E}^{r+1}$.*

This means that for any sections ∇ of $\mathrm{Aff}(\overline{E}^{r+1})$ and θ of $\overline{E}^{r+1}$, there is a well-defined section $\nabla+\theta$ of $\mathrm{Aff}(\overline{E}^{r+1})$; here, addition is defined by

$$[\xi, \Gamma + W] + [\xi, \eta + W] = [\xi, \Gamma + \eta + W]. \tag{38}$$

Given sections ∇_1, ∇_2 of $\mathrm{Aff}(\overline{E}^{r+1})$, the unique section θ such that $\nabla_2 = \nabla_1 + \theta$ is naturally written $\theta = \nabla_2 - \nabla_1$.

We also need to know how automorphisms of $(M,\mathcal{F})$ act on sections of $\mathrm{Aff}(\overline{E}^{r+1})$ or $\overline{E}^{r+1}$. Let f be a smooth automorphism of $(M,\mathcal{F})$ which preserves the structure Q. This means that the automorphism of $F^{r+1}(M,\mathcal{F})$ defined by $f^{(r+1)}(j^{r+1}\varphi_0) = j^{r+1}(f\circ\varphi)_0$ maps Q to itself. Now, for each element $[\xi,\kappa]$ either of $(Q\times\overline{W})/H'$ or $(Q\times\overline{W})/\sim_\rho$, define $(f_*)_x[\xi,\kappa] = [f^{(r+1)}(\xi),\kappa]$, where x is the base point of ξ. Notice that for $[\xi,\kappa]\in\overline{E}^{r+1}$ (the latter quotient), this action coincides with the ordinary push-forward of tensors on $T\mathcal{F}$. In particular, it only depends on df_x.

If ∇ is a section of $\mathrm{Aff}(\overline{E}^{r+1})$, the *push-forward* of ∇ under f is the section $f_*\nabla$ defined by $(f_*\nabla)_x := (f_*)_{f^{-1}(x)}\nabla_{f^{-1}(x)}$. The corresponding definition for sections of $\overline{E}^{r+1}$ gives the ordinary push-forward of tensor fields, and we have: $f_*\nabla_2 - f_*\nabla_1 = f_*\theta$. The *pull-back* of a τ under f is $f^*\tau := (f^{-1})_*\tau$.

5. THE INVARIANT $H_{\mathcal{V}}$-STRUCTURE OF A CONTRACTION

Let $\mathcal{F}$ be a continuous lamination by smooth manifolds of a compact metric space M. Suppose that $T\mathcal{F}$ is equipped with a continuous Riemannian metric and denote by $\|\cdot\|_x$ the associated norm on $T_x\mathcal{F}$. This is needed to define exponential growth rate of vectors under the action of automorphisms of $\mathcal{F}$ and to define a contracting map. Let f be a smooth contracting map of $\mathcal{F}$ whose associated spectral partition $\mathcal{V}$ satisfies the narrow band spectrum condition.

We prove now the normal forms theorem in infinitesimal form. In other words, we obtain the invariant $H_{\mathcal{V}}$-reductions $P^r_{\mathcal{V}}$ of $F^r(M,\mathcal{V})$. As will be seen below, the inductive argument produces $P^r_{\mathcal{V}}$ from $P^{r-1}_{\mathcal{V}}$ using the classical 'geometric sum' method of hyperbolic dynamics to solve (for Ψ) a cohomological equation of the form $\Theta = \Psi - f^*\Psi$, where Θ is a section of a certain tensor bundle over $(M,\mathcal{F})$ defined in the proof.

Let $H^r_{\mathcal{V}}$ denote the projection of $H_{\mathcal{V}}$ into $G^r(V)$, for $r = 1,2,\cdots$. Due to Lemma 7, $H^r_{\mathcal{V}} = H_{\mathcal{V}}$ for all $r \geq d_{\mathcal{V}}$. Notice that $H^1_{\mathcal{V}}$ is the subgroup of $GL(V)$ that stabilizes the flag $W_1 \subset \cdots \subset W_l$ where $W_i = V_1 \oplus \cdots \oplus V_i$, $i = 1,\cdots,l$. The flag $F_1 := E_1 \subset \cdots \subset F_i := E_i \oplus \cdots \oplus E_1 \subset \cdots \subset F_l := T\mathcal{F}$ defines an f-invariant $H^1_{\mathcal{V}}$-reduction of $F^1(M,\mathcal{F})$. Explicitly, this is the principal $H^1_{\mathcal{V}}$-bundle $P^1_{\mathcal{V}} \subset F^1(M,\mathcal{V})$ whose fiber over $x \in M$ consists of all linear isomorphisms $\sigma : V \to T_x\mathcal{F}$ that restrict to isomorphisms from W_i to $F_i(x)$, for each i. We say that an $H_{\mathcal{V}}$-structure on $(M,\mathcal{F})$ is *adapted* to the flag F_i if the $H^1_{\mathcal{V}}$-reduction of $F^1(M,\mathcal{F})$ (obtained by jet projection) is this flag.

Proposition 11. *Let f be a smooth automorphism of the continuous lamination $(M,\mathcal{F})$. Let n be the leaf dimension of $\mathcal{F}$ and $\mathcal{V}$ the spectral partition of $V = \mathbb{R}^n$ associated to f. Suppose that f is a contraction that satisfies the narrow band condition. Then, for every integer $r \geq d_{\mathcal{V}}$ there exists a continuous, f-invariant $H_{\mathcal{V}}$-reduction, $P^r_{\mathcal{V}}$, of $F^r(M,\mathcal{F})$ adapted to the flag $\{F_i\}$. Furthermore,*

1. *$P^r_{\mathcal{V}}$ is the unique f-invariant $H_{\mathcal{V}}$-reduction of $F^r(M,\mathcal{F})$ adapted to $\{F_i\}$;*
2. *if g is an automorphism of $(M,\mathcal{F})$ that commutes with f, then $P^r_{\mathcal{V}}$ is also g-invariant;*
3. *if the flag $\{F_i\}$ is C^k, $k \geq 0$, then $P^r_{\mathcal{V}}$ is also C^k.*

Proof. We only carry out the proofs of parts 1 and 2. Part 3 also follows by a simple adaptation of the argument of [Gu], but we omit the details. (See [Fe1] for the argument in the particular case of invariant first order connections.)

The proof will develop by induction on r. The flag $\{F_i\}$ provides the first step of the induction argument; that is, it corresponds to an f-invariant $H^1_{\mathcal{V}}$- reduction of $F^1(M,\mathcal{F})$.

Suppose that we have obtained a continuous f-invariant $H^s_{\mathcal{V}}$-reduction, $P^s_{\mathcal{V}} \subset F^s(M,\mathcal{F})$, for a $s \geq 1$. Define $H' := (\pi^{s+1}_s)^{-1}(H^s_{\mathcal{V}})$. We seek to find a continuous f-invariant $H^{s+1}_{\mathcal{V}}$-reduction, $P^{s+1}_{\mathcal{V}}$, of the H'-subbundle

$$Q := (\pi^{s+1}_s)^{-1}(P^s_{\mathcal{V}}) \subset F^{s+1}(M,\mathcal{F}). \tag{39}$$

As discussed in Section 4.4, this amounts to finding a continuous f-invariant section of the affine bundle $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}}) = (Q \times \overline{W})/H'$, where $\overline{W} = (S_{s+1}(V^*) \otimes V)/W^{s+1}_{\mathcal{V}}$ and $W^{s+1}_{\mathcal{V}}$ is the ($H^1_{\mathcal{V}}$-invariant) subspace of $S_{s+1}(V^*) \otimes V$ such that $\eta \in W^{s+1}_{\mathcal{V}}$ iff $(I,0,\cdots,0,\eta)$ lies in $\ker(\pi^{s+1}_s|_{H^{s+1}_{\mathcal{V}}})$. Recall that $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}})$ is an affine bundle modeled on the vector bundle $\overline{E}^{s+1}_{\mathcal{V}} = (Q \times \overline{W})/\sim_\rho$, as defined in that section. Also recall that $W^{s+1}_{\mathcal{V}}$ is the direct sum of the subspaces $V^i_J := V^*_{j_1} \otimes \cdots \otimes V^*_{j_{s+1}} \otimes V_i$ for which $\lambda_{j_1}+\cdots+\lambda_{j_{s+1}}\geq\lambda_i$. Observe that $\overline{W}$ is canonically isomorphic to the linear subspace of $S_{s+1}(V^*)\otimes V$ spanned by the V^i_J for which $\lambda_{j_1}+\cdots+\lambda_{j_{s+1}}<\lambda_i$. Therefore,

$$\overline{E}^{s+1}_{\mathcal{V}}=\bigoplus_{i,J}E^i_J,\ \text{summed over } i,J \text{ such that } \lambda_{j_1}+\cdots+\lambda_{j_{s+1}}<\lambda_i. \tag{40}$$

The f-invariant section of $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}})$ is now obtained as follows. We start with an arbitrary continuous section, ∇, of $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}})$. Such a section exists since this bundle has contractible fiber. Now define a family of sections, $\Theta(n)$, $n=1,2,\cdots$, of $\overline{E}^{s+1}_{\mathcal{V}}$ by

$$\Theta(n):=(f^n)^*\nabla-\nabla. \tag{41}$$

It is immediate from this definition that the following *cocycle identity* holds:

$$\Theta(n+m)=\Theta(n)+(f^n)^*\Theta(m) \tag{42}$$

for all $n,m\in\mathbb{Z}$. We wish to show that $\Theta(1)$ is a *coboundary*; that is, show that for a continuous section Ψ of $\overline{E}^{s+1}_{\mathcal{V}}$

$$\Theta=\Psi-f^*\Psi, \tag{43}$$

where $\Theta:=\Theta(1)$. Once a Ψ is obtained, the theorem results from the observation that $\nabla':=\nabla+\Psi$ is the f-invariant continuous section of $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}})$ we seek.

To find Ψ, first note that iterating the coboundary equation yields:

$$\Psi=\Theta+f^*\Theta+\cdots+f^{n*}\Theta+f^{n+1^*}\Psi. \tag{44}$$

This suggests looking for a solution Ψ of the form

$$\Psi:=\sum_{i=0}^{\infty}f^{i^*}\Theta. \tag{45}$$

Therefore, all that we need is show that this series converges to a continuous section. Let Θ^j_I denote the component of Θ in E^j_I. This can only be nonzero if $\lambda_{i_1}+\cdots+\lambda_{i_{s+1}}<\lambda_j$, which implies (this is the essence of Lemma 7):

$$b_{i_1}+\cdots+b_{i_{s+1}}<a_j. \tag{46}$$

On the other hand, by a simple calculation (essentially that used in part 5 of Lemma 7), there exists a constant $C>0$ such that

$$\|f^{n*}\Theta^j_I\|\leq C\|\Theta^j_I\|e^{n(b_{i_1}+\cdots+b_{i_{s+1}}-a_j)}, \tag{47}$$

which decreases exponentially as $n\to\infty$. Therefore, Ψ exists by elementary facts about infinite series.

To prove uniqueness, it suffices to show that the section ∇ obtained at each step of the induction argument is unique. If ∇_1,∇_2 are two f-invariant continuous sections of $\mathrm{Aff}(\overline{E}^{s+1}_{\mathcal{V}})$, the difference $\nabla_2-\nabla_1$ is an f-invariant continuous section of $\overline{E}^{s+1}_{\mathcal{V}}$. But the same estimation used above shows that there is a constant $C>0$ such that $\|\nabla_2-\nabla_1\|=\|f^{n*}(\nabla_2-\nabla_1)\|\to 0$ as $n\to\infty$.

If g commutes with f, then g preserves the flag $\{F_i\}$, so that $P^1_{\mathcal{V}}$ is g-invariant. At each induction step, uniqueness of the f-invariant section ∇ implies that $g*\nabla=\nabla$. Therefore, $P^r_{\mathcal{V}}$ is g-invariant for all r. □

5.1. **Proof of Theorem 6.** Part 1 of Theorem 6 is immediate from the comments made prior to the statement of the theorem. For parts 2 and 3, we first set some notation.

Let $F^r(M,\mathcal{F})/H_{\mathcal{V}}$ denote the associated bundle over $(M,\mathcal{F})$ with fiber $H_{\mathcal{V}}\backslash G^r$, defined by the right-action of G^r on $H_{\mathcal{V}}\backslash G^r$. Let $p : F^r(M,\mathcal{F}) \to F^r(M,\mathcal{F})/H_{\mathcal{V}}$ stand for the natural projection. If E is a fiber bundle, we denote by J^sE the space of s-jets of (germs) of sections of E. The spaces $J^sF^r(V)_0$ and $J^s(F^r(M,\mathcal{F})/H_{\mathcal{V}})$, in particular, are needed below. The former indicates the s-jets at 0 of sections of $F^r(V)$, where V, as before, stands for $\mathbb{R}^n$.

Central to the proof is the following remark. It is a general fact about jet bundles that an element σ of $F^{r+s}(M,\mathcal{F})$ determines canonically an element of $J^sF^r(M,\mathcal{F})$ whose 0-jet is $\pi^{r+s}_r(\sigma)$. Now, if σ belongs to $P^{r+s}_{\mathcal{V}}$, the associated element of $J^sF^r(M,\mathcal{F})$ actually lies in $J^sP^r_{\mathcal{V}}$. This is what is proved next. Notice that the proof is not only formal jet theory as invariance under the contracting map f is used in an important way.

Denote by t_x the translation map of V, so that $t_x(z) = z + x$, $z \in V$. Notice that, for any given $\psi \in H_{\mathcal{V}}$ and $x \in V$, the diffeomorphism $\psi_x := t_{-\psi(x)} \circ \psi \circ t_x$ also lies in $H_{\mathcal{V}}$. In fact, it is easy to obtain

$$\lim_{n\to\infty} T^n_{\mathcal{V}*}\psi_x = \sum_{i=1}^{l} \pi_i\psi_1\pi_i, \tag{48}$$

where ψ_1 is the linear part of ψ and π_i denotes the projection from V to V_i.

Define $\tau(x)$, $x \in V$, as the r-jet at 0 of t_x. The r-jet at 0 of the identity map of V will be written e_r, or simply e if no confusion may arise. Of course, $e = \tau(0)$. We call τ the canonical section of $F^r(V)$.

To each $\sigma \in F^{r+s}(M,\mathcal{F})$, where $\sigma = j^{r+s}\phi_0$, we can associate the map $\phi^{(r)} : F^r(V) \to F^r(M,\mathcal{F})$ defined by $\phi^{(r)}(j^r\phi'_0) = j^r(\phi\circ\phi')_0$. This is well-defined up to jet of order s at e. Then $\phi^{(r)}$ is a bundle map in that $\phi^{(r)}(\xi g) = \phi^{(r)}(\xi)g$ for all $\xi \in F^r(V)$ and all $g \in G^r$. Similarly, $h = j^{r+s}\psi_0 \in G^{r+s}$ defines a map $\psi^{(r)} : F^r(V) \to F^r(V)$. A simple consequence of these elementary facts and definitions is that the following equation holds for each $x \in V$:

$$\phi^{(r)}(\psi^{(r)}(\tau(x))) = \phi^{(r)}(\tau(\psi(x)))\psi_x. \tag{49}$$

To σ we associate an element $\mathcal{S}(\sigma)$ of $J^s(F^r(M,\mathcal{F})/H_{\mathcal{V}})_{p(\bar\sigma)}$, where $\bar\sigma = \pi^{r+s}_r(\sigma)$, by setting

$$\mathcal{S}(\sigma) := j^s(p\circ\phi^{(r)}\circ\tau\circ\phi^{-1})_{\phi(0)}. \tag{50}$$

Observe that $\mathcal{S}(\sigma)$ does not depend on the choice of ϕ defining σ. By the next lemma, it also does not depend on the choice of σ on a fiber of $P^{r+s}_{\mathcal{V}}$.

Lemma 12. *If $\sigma \in F^{r+s}(M,\mathcal{F})$ and $h \in H_{\mathcal{V}}$, then $\mathcal{S}(\sigma h) = \mathcal{S}(\sigma)$.*

Proof. Write $h = (j^{r+s}\psi)_0$ and $\sigma = (j^{r+s}\phi)_0$. Recall that ψ_x also belongs to $H_{\mathcal{V}}$. So it follows from Equation 49 that $p\circ\phi^{(r)}\circ\psi^{(r)}\circ\tau = p\circ\phi^{(r)}\circ\tau\circ\psi$. Composing on the right with $(\phi\circ\psi)^{-1}$, we obtain:

$$p\circ(\phi\circ\psi)^{(r)}\circ\tau\circ(\phi\circ\psi)^{-1} = p\circ\phi^{(r)}\circ\tau\circ\phi^{-1}. \tag{51}$$

Taking s-jets at $\phi(0)$ yields the claim. □

Let $\mathcal{P}$ denote (the image of) the section of the bundle $F^r(M,\mathcal{F})/H_{\mathcal{V}}$ associated to the $H_{\mathcal{V}}$-reduction $P^r_{\mathcal{V}}$. (Recall that sections of that bundle correspond bijectively with $H_{\mathcal{V}}$-reductions of $F^r(M,\mathcal{V})$.) Thus $\mathcal{P}$ is homeomorphic to M via the base point projection of $F^r(M,\mathcal{F})/H_{\mathcal{V}}$. By the above lemma we obtain a continuous map $[\sigma] \in \mathcal{P} \mapsto \mathcal{S}([\sigma])$. The group of smooth automorphisms of $(M,\mathcal{F})$ naturally acts on $F^r(M,\mathcal{F})/H_{\mathcal{V}}$, and those automorphisms that leave $P^r_{\mathcal{V}}$ invariant as a set will act on that associated bundle so as to leave the set $\mathcal{P}$ invariant. If in addition the automorphism leaves $P^{r+1}_{\mathcal{V}}$ invariant, then $\mathcal{S}$ is an invariant function on $\mathcal{P}$.

We now take into account the assumption that $P^r_{\mathcal{V}}$, hence $\mathcal{P}$, is C^1. To prove that the quasi-connection η^r is a generalized connection, we need to verify that it is tangent to $P^r_{\mathcal{V}}$. Equivalently, we need to show that $\mathcal{S}([\sigma])$ (for $s=1$) is tangent to $\mathcal{P}$ at $[\sigma]$ for each $\sigma \in P^r_{\mathcal{V}}$. We think of $\mathcal{S}$ and the tangent bundle $T\mathcal{P}$ as n-dimensional subbundles of the tangent bundle of $F^r(M,\mathcal{F})/H_{\mathcal{V}}$ restricted (pulled-back) to $\mathcal{P}$. Both are invariant under the map which the contraction induces on $F^r(M,\mathcal{F})/H_{\mathcal{V}}$. We denote the map by f.

On $T\mathcal{P}$, f acts as a contraction, just as it does on $T\mathcal{F}$. The fibers of $F^r(M,\mathcal{F})/H_{\mathcal{V}}$ can be identified with the quotient $G^r/H_{\mathcal{V}}$. By what we saw in the proof of Proposition 11, f^{-1} acts on the fibers of $F^r(M,\mathcal{F})/H_{\mathcal{V}}$ also as a contraction. From this and the fact that $\mathcal{S}$ does not contain vertical vectors it follows that the invariant subbundles $\mathcal{S}$ and $T\mathcal{P}$ must coincide. But $\mathcal{S}$ is precisely the image under the differential of p of the connection n-plane η, which means that η must be contained in $TP^r_{\mathcal{V}}$. Therefore, η is a generalized connection.

For η to be a (principal) connection, it needs to be $H_{\mathcal{V}}$-invariant. We now check that this is the case when $r=1$. Let $\sigma \in F^{r+1}(M,\mathcal{F})$, and consider the linear map

$$\mathcal{H}_\sigma : V \to T_{\bar\sigma}F^r(M,\mathcal{F}) \tag{52}$$

given by $\mathcal{H}_\sigma := (d\phi^r)_e \circ (d\tau)_0$.

For x on a sufficiently small neighborhood of $0\in V$ and $\psi \in H_{\mathcal{V}}$, define $\tilde h(x)$ as the r-jet at $0\in V$ of the local diffeomorphism, $z \mapsto (t_{-\psi^{-1}(x)} \circ \psi^{-1} \circ t_x \circ \psi)(z)$, of V. Then, as we saw before, $\tilde h(x) \in H^r_{\mathcal{V}}$, for each x near $0 \in V$. Since $\tilde h(0)$ is the identity in $H^{\mathcal{V}}$, we obtain a linear map, $d\tilde h_0$, from V to the Lie algebra $\mathfrak{h}_{\mathcal{V}}$ of $H_{\mathcal{V}}$. Now, for each $\bar\sigma \in P^r_{\mathcal{V}}$ and $h \in H_{\mathcal{V}}$, define a linear map

$$\mathfrak{H}_{\bar\sigma}(h) : V \to T_{\bar\sigma}F^r(M,\mathcal{F}) \tag{53}$$

which associates to $v \in V$ the value at $\bar\sigma$ of the standard vertical vector field associated to $d\tilde h_0 v$. Observe that $\tilde h$ is constant when $h \in H_{\mathcal{V}}$ is linear, so $\mathfrak{H}_\sigma(h) = 0$ in this case.

It is now not difficult to show that

$$(dR_h)_\sigma \circ \mathcal{H}_{\xi(\sigma)} = \mathcal{H}_{\xi(\sigma)h} \circ A^{-1} + \mathfrak{H}_{\sigma h}(h) \tag{54}$$

holds for each $h \in H_{\mathcal{V}}$, where $A = \pi^{r+1}_1(h)$. When $H_{\mathcal{V}}$ is a subgroup of $GL(V)$ (as in part 3 of Theorem 6), $\mathfrak{H}$ vanishes as noted above. Therefore, the generalized connection η, which is the image of V under $\mathcal{H}$, is $H_{\mathcal{V}}$-invariant, hence a connection.

Flatness of the generalized connection (i.e., involutiveness of the n-plane distribution η) is due to the fact that $P^{r+1}_{\mathcal{V}}$ defines, in the language of [Gr], a 'complete' and 'consistent' partial differential relation, so that the classical Frobenius theorem applies. (The property of 'completeness' is related to $\pi^{r+1}_r : P^{r+1}_{\mathcal{V}} \to P^r_{\mathcal{V}}$ being injective, and 'consistency' comes from $\pi^{r+2}_r : P^{r+2}_{\mathcal{V}} \to P^{r+1}_{\mathcal{V}}$ being surjective.) A detailed discussion of this is found in [Fe3], in particular Lemma 5.3.

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