Math 308 - Exam III - April 12, 2002

The test contains 8 questions, each of equal value. Whenever possible, answers should be written using exact numbers. For example: write $\frac{2}{3}$ instead of 0.67, π instead of 3.1415, e^2 instead of 7.4, etc.

1. Evaluate the integral $\iint (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, d\sigma$ over the surface consisting of the four slanting faces of a pyramid whose base is the square in the (x, y) plane with corners at $(0,0), (0,2), (2,0), (2,2)$ and the top vertex is at $(1,1,2)$, where $\mathbf{V} = (x^2z - 2)\mathbf{i} + (x + y - z)\mathbf{j} - xyz\mathbf{k}$.

2. Evaluate $\iint \mathbf{V} \cdot \mathbf{n}\, d\sigma$ over the entire surface of the sphere of radius 3 and center $(2, -3, 0)$ if $\mathbf{V} = (3x - yz)\mathbf{i} + (z^2 - y^2)\mathbf{j} + (2yz + x^2)\mathbf{k}$. (Suppose that $\mathbf{n}$ is oriented outward.)

3. Let $\mathcal{R}$ be a region in the xy-plane whose boundary, $\partial\mathcal{R}$, is a simple closed curve. Use Stokes' theorem to show that
$$\text{Area}(\mathcal{R}) = \frac{1}{2}\oint_{\partial\mathcal{R}} \mathbf{V} \cdot d\mathbf{r}$$
where $\mathbf{V}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

4. Find the complex Fourier series of a periodic function $f(x)$ of period 2 whose values over the interval $[0, 2)$ are as follows:
$$f(x) = \begin{cases} 3 & \text{for } 0 \le x < 1 \\ 1 & \text{for } 1 \le x < 2. \end{cases}$$

5. The *Dirac delta-function*, $\delta_{x_0}(x)$, is a (generalized) function, defined by the following property: for every continuous function $h(x)$, and every interval $[a, b]$ such that $a < x_0 < b$,
$$\int_a^b h(x)\delta_{x_0}(x)dx = h(x_0).$$
Physically, $\delta_{x_0}(x)$ can be thought of as the density of a point mass (of mass $m = 1$) that is entirely concentrated at the point x_0.

   Find the sine and the cosine Fourier series of $\delta_{l/2}(x)$ over the interval $[0, l]$.

6. A bar of length l with insulated sides has an initial distribution of temperature given by a Dirac delta-function concentrated at the middle point. Starting at $t = 0$, the ends are held at $0°$. Find the distribution of temperature in the bar at time t. (The heat equation is $u_t = \alpha^2 u_{xx}$.

7. Find the cosine Fourier series of the function $f(x) = x$ over the interval $0 \le x \le l$.

8. A bar of length l with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature is $u = x$, where x is the distance from one end. The temperature distribution inside the bar at time t is
$$u(t, x) = A + \sum_{\text{odd } n} B_n \cos\frac{n\pi x}{l} e^{-(n\pi\alpha/l)^2 t}$$
Determine the constants A and B_n.