Math 308 - Final Exam - April 26, 2002

The test contains 8 questions, each of equal value. Whenever possible, answers should be written using exact numbers. For example: write $\frac{2}{3}$ instead of 0.67, π instead of 3.1415, e^2 instead of 7.4, etc.

1. Answer the following questions about the matrix

$$A = \begin{pmatrix} 1 & 0 & a \\ 0 & 2 & b \\ 0 & 1 & 0 \end{pmatrix}.$$

   (a) For what values of a and b does the inverse matrix exist?

   (b) Find the inverse A^{-1}.

   (c) Explain how the inverse matrix A^{-1} can be used to find the solution of the linear system:

$$\begin{aligned} x + az &= 1 \\ 2y + bz &= 0 \\ y &= 1. \end{aligned}$$

2. Given the function $\phi = z \sin y - xz$,

   (a) find the gradient of ϕ at the point $(2, \pi/2, -1)$.

   (b) Starting at the point of part (a), in what direction is ϕ decreasing most rapidly?

   (c) What is the directional derivative of ϕ in the direction $2\mathbf{i} + 3\mathbf{j}$? (Do not forget to normalize the direction vector.)

3. Let $\mathbf{F}(x, y, z) = (-y\mathbf{i} + x\mathbf{j})/(x^2 + y^2) + 3\mathbf{k}$ be a vector field in $\mathbb{R}^3$.

   (a) Find $\nabla \times \mathbf{F}$.

   (b) Find $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle of radius 1, center $(0, 0, 1)$, traversed counterclockwise (viewed from above, looking down on the plane of the circle).

   (c) Explain why the following is true: If D is any closed curve that circles once around the z-axis in the counterclockwise direction, then $\int_D \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the circle from part (a). (Make sure to state which important theorem is needed, and how it is used to derived the claim.)

4. Let $\mathbf{F} = ((x-1)\mathbf{i} + (y-2)\mathbf{j} + (z+1)\mathbf{k})/((x-1)^2 + (y-2)^2 + (z+1)^2)^{3/2}$.

   (a) Find $\nabla \cdot \mathbf{F}$.

   (b) Find $\iint_S \mathbf{F} \cdot \mathbf{n} d\sigma$, where S is the sphere of radius 1, center $(1, 2, -1)$, oriented so that $\mathbf{n}$ points outward.

   (c) Explain why the integral of item (b) is equal to $\iint_R \mathbf{F} \cdot \mathbf{n} d\sigma$ where R is the surface of a sphere of radius 10, center $(0, 0, 0)$. In particular, what important theorem is needed here?

5. Decide whether the following claims are true or false and give a brief explanation in each case.

   (a) Every gradient vector field has the path-independence property.

   (b) If $\mathbf{V}$ is an everywhere differentiable vector field, the flux of $\nabla \times \mathbf{V}$ across a closed surface is always zero.

   (c) Every surface can be given two different orientations.

   (d) Even if a vector field $\mathbf{V}$ has zero curl in a region $\mathcal{R}$, the integral $\int_C \mathbf{V} \cdot d\mathbf{R}$ need not be zero for every closed path C in $\mathcal{R}$.

6. Let $f(x)$ be a periodic function of period 2 such that $f(x) = x^2$ over the interval $-1 \leq x \leq 1$.

   (a) Find the complex exponential Fourier series of $f(x)$.

   (b) Using the result of part (a), obtain the real series (in sines and cosines).

7. A bar of length 2 is initially at $0°$. From $t = 0$ on, the end $x = 0$ is held at $0°$ and the end $x = 2$ at $100°$. Find the time-dependent temperature distribution.

8. Maxwell's equations in free space take the form

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

   where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields. Show that each component of $\mathbf{E}$ and $\mathbf{H}$ satisfies a wave equation. Identify the wave velocity in terms of the constants μ and ϵ.