Math 308 - Exam 1 - February 7, 2003

The test contains 8 questions, each of equal value. Whenever possible, answers should be written using exact numbers. For example: write $\frac{2}{3}$ instead of 0.67, π instead of 3.1415, e^2 instead of 7.4, etc.

1. Find the gradient of $\phi = x \sin z + 2yz$ at the point $(2, 1, \pi)$. Starting at this point, find the directional derivative of ϕ in the direction
$$\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{k}.$$

2. Compute the divergence and curl of the vector field
$$\mathbf{V} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}.$$

3. Calculate the Laplacian ∇^2 of $x^3 - 3xy^2 + y^3$.

4. Which of the following two vector fields is conservative? For the one which is, find a scalar function ϕ such that $\mathbf{V} = -\nabla\phi$.
   (a) $\mathbf{V} = (3x^2yz - 3y)\mathbf{i} + (x^3z - 3x)\mathbf{j} + (x^3y + 2z)\mathbf{k}$
   (b) $\mathbf{V} = (3x^2yz - 3y)\mathbf{i} + (x^3z - 3x)\mathbf{j} - (x^3y + 2z)\mathbf{k}$

5. For the gradient force field $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ calculate the work done in moving a particle from $(0, 0, 0)$ to $(\pi, -\pi, \pi^3)$ along the curve
$$x = t, \quad y = t\cos^3 t, \quad z = t^3 \sin(t/2).$$

6. Evaluate the integral $\oint (x + 2y)dx - 2xdy$ along the circle $x^2 + y^2 = 1$ taken counterclockwise.

7. Evaluate $\iint (\nabla \times \mathbf{V}) \cdot \mathbf{n}\, d\sigma$ over the surface $z = 9 - x^2 - y^2$, $z \geq 0$, where
$$\mathbf{V} = (x - x^2z)\mathbf{i} + (yz^3 - y^2)\mathbf{j} + (x^2y - xz)\mathbf{k}.$$

8. Evaluate $\iint \mathbf{V} \cdot \mathbf{n}\, d\sigma$, where $\mathbf{V} = y\mathbf{i} + xz\mathbf{j} + (2z - 1)\mathbf{k}$, over the *curved* surface of the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$.