Math 308 - Exam III - Date due: April 17, 2002

1 Ground temperature variation

In this problem-set you will do a 'back of an envelope' calculation to study temperature variations of the ground at various depths due to the sun's heating. The original study is due to Fourier, Poisson and Kelvin.

The model treats the earth as a half-space $\{(x, y, z) \in \mathbb{R}^3 | x \geq 0\}$, where x represents depth and y, z are local coordinates for the earth's surface. It will be assumed that all parameters depend only on depth, thus making the problem effectively one-dimensional (y and z will not appear). It is also assumed that values of x are not so large as to require taking into account sources of heat in the interior of the earth.

Write $\Theta(t, x)$ (*theta*) for the temperature at depth x and time t. The 1-dimensional heat equation reads:

$$\frac{\partial \Theta}{\partial t}(t, x) = K \frac{\partial^2 \Theta}{\partial x^2}(t, x), \tag{1}$$

where the constant K only depends on the heat conducting material. For ordinary soil, take $K = 2 \times 10^{-3}\text{cm}^2/\text{sec}$. The goal will be to obtain the phase lag (i.e., the time shift) of temperature variations at a depth x compared to that at the surface, and the decrease in the amplitude of the variation, both diurnal and annual. In particular, at what depth are the seasons inverted, so that one has, say, Winter when it is Summer at surface level?

To equation (1) we add the (time-dependent) boundary condition

$$\Theta(t, 0) = f(t), \tag{2}$$

where $f(t)$ is a periodic function of period τ, which could equal 24 hours if we are concerned with daily temperature variation, or 365 days (for annual variation). No initial condition will be imposed. We will take the unit of time to be 1 day. A simple class of functions that takes into account both daily and annual variations in surface temperature might be

$$f(t) = T_0 + T_1 \cos \frac{2\pi t}{365} + T_2 \cos 2\pi t, \tag{3}$$

for suitable choices of T_0, T_1, T_2.

We also impose the (physically plausible!) condition that Θ is a bounded function, that is, for some big enough constant C,

$$|\Theta(t, x)| \leq C \text{ for all } t, x. \tag{4}$$

Your work will be broken up into a number of problems, enumerated below.

2 Problems

1. Choose T_0, T_1, T_2 (in Fahrenheit) for which $f(t)$ is, in your opinion, a reasonable representation of the variation of surface temperature along the year in a place like St. Louis. Explain your choice of constants. (No need to research this. Your gut feeling should be good enough.) Also draw a rough sketch of the graph of f to help see the relative amplitudes of daily and annual temperature oscillations.

2. Assume for the moment that $f(t)$ is an arbitrary periodic function of period τ, where $\tau = 365$, and that the solution takes the following form:

$$\Theta(t, x) = \sum_{n=-\infty}^{\infty} U_n(x) e^{2\pi i n t/\tau}. \qquad (5)$$

   Find a second order ordinary differential equation for $U_n(x)$ so that Θ is a solution of equation 1. What should $U_n(0)$ be if we impose the initial condition given by 2? (At one point in your calculation you will need to express $\sqrt{i}$ and $\sqrt{-i}$ in the form $a + ib$. Also note that because of the boundedness condition 4 we should discard exponentially increasing terms.)

3. Still assuming that $f(t)$ is a general τ-periodic function, find $U_n(x)$. (You will notice that the ODE you get in the previous item, plus the value of $U_n(0)$ and the boundedness condition (4) uniquely determine $U_n(x)$.)

4. Find Θ for $f(t)$ as in equation (3). (Write your solution for general constants T_0, T_1, T_2, rather than the particular choice you made in problem 1.)

5. If we are concerned about the annual variation of temperature, the rapidly oscillating term $T_2 \cos 2\pi t$ can be "averaged out" so that T_2 may be set equal to 0. Suppose this is case. Find the depth x (nearest to the surface) at which the temperature is completely out of step with that on ground level (so that one reaches a maximum when the other reaches a minimum). At that depth, how does the amplitude of the temperature oscillation compares with the amplitude of the temperature oscillation on the surface? (Does it oscillate more or less widely? by how much?) Express your answer in meters, and use your estimates for T_0 and T_1. (Note: If you write your solution in the form

$$\Theta(t, x) = F(t) \cos\left(\frac{2\pi t}{\tau} - g(x)\right), \qquad (6)$$

   you can find the phase shift by determining $g(x)$.)