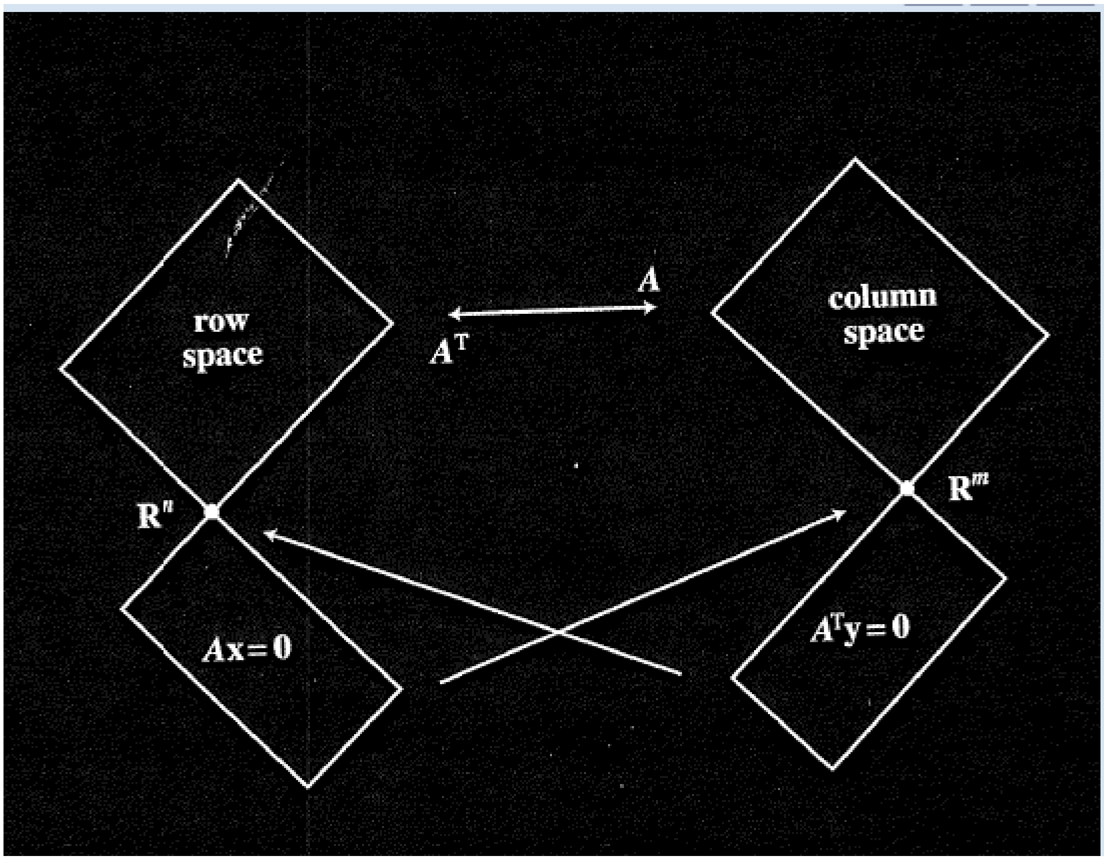
The four "fundamental subspaces associated with an $m \times n$ matrix A
Nul A, Row A, Col A, Nul A^T

The mapping $\boldsymbol{x} \longrightarrow A\boldsymbol{x}$
goes from $\mathbb{R}^n$ to $\mathbb{R}^m$



$A^T\boldsymbol{y} \longleftarrow \boldsymbol{y}$
This mapping goes from $\mathbb{R}^m$ to $\mathbb{R}^n$

In $\mathbb{R}^n$ we have subspaces

In $\mathbb{R}^m$ we have

not the same, but <u>both</u> have

Row $\boldsymbol{A}$ $\longleftarrow$ **dim $= \boldsymbol{r}$ (rank A)** $\longrightarrow$ **Row $\boldsymbol{A}^T$ $=$ Col $\boldsymbol{A}$**

Nul $\boldsymbol{A}$, with dim $= n - r$ | **Nul $\boldsymbol{A}^T$**, with dim $= m - r$

Sum of dimensions $= \boldsymbol{n}$ | **Sum of dimensions $= \boldsymbol{m}$**

Each vector in Row A is otrhogonal to each vector in Nul A

Each vector in Row A^T is orthogonal to each vector in Nul A^T.

Example $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}$

rref: $A \sim ... \sim U = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ $A^T \sim ... \sim V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\boldsymbol{x} \longrightarrow A\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix}\boldsymbol{x}$$

$\mathbb{R}^3$ $\mathbb{R}^2$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 2 & 4 \end{bmatrix}\boldsymbol{y} = \boldsymbol{A^T y} \longleftarrow \boldsymbol{y}$$

Row A : Basis $\{(1,0,2), \ (0,1,2)\}$
dimension 2

Nul A: Basis $\left\{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}\right\}$

dimension 1

Sum of dimensions $= n = 3$

Row A^T: Basis $\{(1,0), \ (0,1)\}$
||
Col A : Basis $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

dimension 2

Nul $A^T = \{\mathbf{0}\}$ (no basis)

dimension $= 0$

Sum of dimension $= m = 2$

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$$

This forces each vector in Row A to be orthogonal to each vector in Nul A

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This forces each vector in Row A^T to be orthogonal to each vector in Nul A^T