Additional Example Let $A = \begin{bmatrix} 1 & -3 & 4 & 1 \\ 2 & 1 & 1 & -1 \\ 1 & -1 & -1 & 2 \end{bmatrix}$ and consider the systems

$$A\boldsymbol{x} = \mathbf{0} \qquad \text{and} \qquad A\boldsymbol{x} = \begin{bmatrix} 12 \\ 4 \\ 7 \end{bmatrix}.$$

Row reducing to solve both systems simultaneously (as in the previous example), we get:

$$\begin{bmatrix} 1 & -3 & 4 & 1 & 0 & 12 \\ 2 & 1 & 1 & -1 & 0 & 4 \\ 1 & -1 & -1 & 2 & 0 & 7 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} \mathbf{1} & 0 & 0 & \frac{1}{3} & 0 & \frac{11}{3} \\ 0 & \mathbf{1} & 0 & -\frac{22}{21} & 0 & -\frac{65}{21} \\ 0 & 0 & \mathbf{1} & -\frac{13}{21} & 0 & -\frac{5}{21} \end{bmatrix}$$

x_1, x_2, x_3 are the basic variables and x_4 is free.

Ths solutions to $A\boldsymbol{x} = \mathbf{0}$ can be written in <u>parametric vector form</u> as

$$\downarrow$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3}x_4 \\ \frac{22}{21}x_4 \\ \frac{13}{21}x_4 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{1}{3} \\ \frac{22}{21} \\ \frac{13}{21} \\ 1 \end{bmatrix} \text{ (with } x_4 \text{ free)}$$

$$\uparrow$$
$$\boldsymbol{v}$$

Every solution is a multiple of the one vector $\boldsymbol{v}$; the set of solutions is a straight line through the origin $\mathbf{0}$ in $\mathbb{R}^4$

One observation for more pleasant notation: <u>replacing</u> $\boldsymbol{v}$ with any <u>nonzero</u> multiple $k\boldsymbol{v}$ works just as well as a way to write down the solutions

This is because $\{t\boldsymbol{v} : t$ is any scalar$\} = \{s(k\boldsymbol{v}) : s$ is any scalar, where k is a <u>fixed</u> nonzero scalar$\}$: each point $s(k\boldsymbol{v})$ on the "second" line is a point $t\boldsymbol{v}$ on the first line (where $t = sk$), and vice versa.

In fact, we might as well write $\{t\boldsymbol{v} : t$ is any scalar$\} = \{t(k\boldsymbol{v}) : t$ is any scalar, where k is a <u>fixed</u> nonzero scalar$\}$

So, in writing the solution in this example, it's prettier to replace $\boldsymbol{v}$ with $21\boldsymbol{v}$:

"the line $t\begin{bmatrix} -\frac{1}{3} \\ \frac{22}{21} \\ \frac{13}{21} \\ 1 \end{bmatrix}$" can also be described as " the line $t\begin{bmatrix} -7 \\ 22 \\ 13 \\ 21 \end{bmatrix}$"

The solutions of $A\boldsymbol{x} = \begin{bmatrix} 12 \\ 4 \\ 7 \end{bmatrix}$ can be read off from the reduced matrix as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{11}{3} - \frac{1}{3}x_4 \\ -\frac{65}{21} + \frac{22}{21}x_4 \\ -\frac{5}{21} + \frac{13}{21}x_4 \\ x_4 \end{bmatrix} = \underset{\substack{\uparrow \\ \boldsymbol{p}}}{\begin{bmatrix} \frac{11}{3} \\ -\frac{65}{21} \\ -\frac{5}{21} \\ 0 \end{bmatrix}} + x_4 \underset{\substack{\uparrow \\ \boldsymbol{v}}}{\begin{bmatrix} -\frac{1}{3} \\ \frac{22}{21} \\ \frac{13}{21} \\ 1 \end{bmatrix}} \quad \text{(with } x_4 \text{ free)}$$

The set of solutions is also a straight line (in $\mathbb{R}^4$) obtained by adding $\boldsymbol{p}$ (translating) the straight line thru the origin which represents the solutions to the homogeneous system $A\boldsymbol{x} = \boldsymbol{b}$.

Note: Here we could also write the solutions as (clearing some fractions and, if you like, changing the parameter from x_4 to s:

$$\boldsymbol{x} = \begin{bmatrix} \frac{11}{3} \\ -\frac{65}{21} \\ -\frac{5}{21} \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 22 \\ 13 \\ 21 \end{bmatrix}$$

CAUTION: explain why you can't clear the other fractions by also multiplying $\begin{bmatrix} \frac{11}{3} \\ -\frac{65}{21} \\ -\frac{5}{21} \\ 0 \end{bmatrix}$ *by* 21.