In the interest of honesty:

Questions 15)-17) all referred to this situation:

Near Deep Space Station K-7, on a small continent in the southern hemisphere of Sherman’s planet, there live 4 different interdependent populations: the Bijks (b), the Ejvaks (e), Ryzgas (r) and the Tribbles (t). With no insult intended to these noble species, we will sometimes refer to their populations by the first letter of their names: the b’s, the e’s, the r’s, and the t’s.

At some time \( t = 0 \), the populations (in thousands) are given by the vector

\[
\mathbf{x}_0 = \begin{bmatrix} b_0 \\ e_0 \\ r_0 \\ t_0 \end{bmatrix} = \begin{bmatrix} 20 \\ 10 \\ 15 \\ 1 \end{bmatrix}.
\]

Each year the populations change according to the equation

\[
\mathbf{x}_{k+1} = A\mathbf{x}_k = \begin{bmatrix} 0.50 & 0 & 0.25 & -0.25 \\ -0.75 & 0.50 & 0.25 & 0.50 \\ -0.75 & 0 & 0.50 & 0.75 \\ -0.75 & 0 & 0.25 & 1.00 \end{bmatrix} \mathbf{x}_k.
\]

We can factor \( A \) as

\[
A = PDP^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 1.25 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}
\]

The intended solution is completely analogous to the examples of dynamic systems with owls and wood rats, or with spotted owls, in section 5.6 (but with no complex numbers involved here).

Since \( A \) is diagonalizable, the columns of \( P = [\mathbf{v}_1 \; \mathbf{v}_2 \; \mathbf{v}_3 \; \mathbf{v}_4] \) are an eigenvector basis for \( \mathbb{R}^4 \), with eigenvalues (from \( D \)) of 0.5, 0.5, 0.25, and 1.25.

We can write \( \mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \), so

\[
\mathbf{x}_1 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3 + c_4A\mathbf{v}_4 = c_1(1.5)\mathbf{v}_1 + c_2(0.5)\mathbf{v}_2 + c_3(0.25)\mathbf{v}_3 + c_4(1.25)\mathbf{v}_4,
\]

\[
\vdots
\]

\[
\mathbf{x}_k = c_1(1.5)^k\mathbf{v}_1 + c_2(0.5)^k\mathbf{v}_2 + c_3(0.25)^k\mathbf{v}_3 + c_4(1.25)^k\mathbf{v}_4
\]

The first 3 terms \( \to 0 \) as \( k \to \infty \), so for large \( k \), we ignore them and approximate

\[
\mathbf{x}_k \approx c_4(1.25)^k\mathbf{v}_4 = c_4(1.25)^k \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_4(1.25)^k \\ c_4(1.25)^k \\ c_4(1.25)^k \end{bmatrix}
\]
So, after many years (i.e., for large $k$): the $b_k \approx 0$ (the $b$'s are dying out) and the ratio
$$\frac{r}{t} \approx \frac{c_4(1.25)^k}{c_4(1.25)^k} = 1,$$
eq etc.

All this can be done without even thinking about the meaning of the $b$'s, $e$'s, $r$'s, and $t$'s: the
“backstory” for the mathematics is just to make the test question seem more colorful.

However:

One student noticed that as you compute $A\mathbf{x}_0 = \mathbf{x}_1$, $A\mathbf{x}_1 = \mathbf{x}_2$, ...
the $b$, $e$, $r$, $t$ values become negative.  (You could also see this in the limit if we had bothered
to compute the weights $c_1$, ..., $c_4$: the weight $c_4$ turns out to be negative, so $c_4(1.25)^k$ is
negative.  (Of course, the ratio $\frac{r}{t}$ is still $\frac{1}{1}$ — no change in the answer!  I didn’t notice this
issue about negative populations because I didn’t actually compute $c_4$ when I wrote the
question, and solving the problem gave me no reason to actually compute $A\mathbf{x}_0 = \mathbf{x}_1$, etc.)

Negative populations of these creatures don't make physical sense, so the colorful backstory
for the mathematics doesn't work -- although the mathematical solution is fine.

You could invent a new story about the mathematics just by imagining $b$, $e$, $r$, $t$ to be
quantities that are allowed to be negative.  For example, we could imagine 4 regions of the
countries B,E,R, and T where (due to climate change) annual temperatures are changing.  We
start with $\mathbf{x}_0 = \begin{bmatrix} b_0 \\ e_0 \\ r_0 \\ t_0 \end{bmatrix}$, where $b_0$ = “average annual temperature in region B” in year 0, etc.

Then $\mathbf{x}_k = \begin{bmatrix} b_k \\ e_k \\ r_k \\ t_k \end{bmatrix}$ would list the average annual temperatures in the 4 regions in year $k$.  In
region B, $b_k \approx 0$ for large $k$: in region B, the average annual temperature is approaching 0.

In regions R and T, the ratio of average annual temperatures $\frac{r}{t}$ is approaching ratio $\frac{1}{1}$, etc.