These are theorems that I proved in class that, I think, cover the material in Section 5.3 much more efficiently. These theorems immediately imply Theorem 5.17-5.29 in the textbook.

**Theorem 1**  If \( A \) is an infinite subset of \( \mathbb{N} \), then \( A \cong \mathbb{N} \).

**Proof**  Since \( A \neq \emptyset \), and \( A \subseteq \mathbb{N} \), the Well-Ordering Principle says there is a smallest member \( a_1 \in A \).

Since \( A \) is not finite, \( A - \{a_1\} \neq \emptyset \). Let \( a_2 \) be the smallest member of \( A - \{a_1\} \).

Since \( A \) is not finite, we can always continue this process another step. We inductively define \( a_{n+1} \) to be the smallest member of \( A - \{a_1, \ldots, a_n\} \).

The sequence \( a_1, a_2, \ldots, a_n, \ldots \) contains every member of \( A \) (why?), and the mapping \( f : A \rightarrow \mathbb{N} \) defined by \( f(a_n) = n \) is one-to-one and onto \( \mathbb{N} \). Therefore \( A \cong \mathbb{N} \).  

**Corollary 2**  A set \( A \) is countable iff there exists a one-to-one function \( f : A \rightarrow \mathbb{N} \) (not necessarily onto!)

**Proof**  If such a function exists, then \( A \cong \text{range}(f) \subseteq \mathbb{N} \). Therefore \( A \) is finite or, by Theorem 1, \( A \cong \mathbb{N} \). Therefore \( A \) is countable.

If \( A \) is countable, then \( A \) is either finite or denumerable so either there is a one-to-one function \( f : A \rightarrow \mathbb{N} \subseteq \mathbb{N} \) or there is a bijection \( f : A \rightarrow \mathbb{N} \). Either way, we have a one-to-one function from \( A \) to \( \mathbb{N} \) so \( A \) is countable.  

**Corollary 3**  A subset of a countable set is countable.

**Proof**  Suppose \( B \subseteq A \) where \( A \) is countable. By Corollary 2, there is a one-to-one function \( f : A \rightarrow \mathbb{N} \). Then \( g = f|B \) is a one-to-one function from \( B \) into \( \mathbb{N} \) so \( B \) is countable.

**Theorem 4**  If \( A \) and \( B \) are countable, then \( A \times B \) is countable.

**Proof**  By assumption, there exist one-to-one functions \( f : A \rightarrow \mathbb{N} \) and \( g : B \rightarrow \mathbb{N} \).

For each ordered pair \( (a, b) \in A \times B \), define \( h(a, b) = 2^{f(a)}3^{g(b)} \in \mathbb{N} \). Then \( h : A \times B \rightarrow \mathbb{N} \).

(Example: if, say, \( f(a) = 27 \) and \( g(b) = 113 \), then \( h((a, b)) = 2^{27}3^{113} \).)

We claim that \( h \) is one-to-one: Suppose \( (a, b) \neq (c, d) \in A \times B \). Then either \( a \neq c \) or \( b \neq d \).

If \( a \neq c \), then \( f(a) \neq f(c) \) since \( f \) is one-to-one. So \( 2^{f(a)}3^{g(b)} \neq 2^{f(c)}3^{g(d)} \) by the Fundamental Theorem of Arithmetic.
If \( b \neq d \), then \( g(b) \neq g(d) \) since \( g \) is one-to-one. So \( 2f(a)3g(b) \neq 2f(c)3g(d) \).

Either way, \( h(a, b) \neq h(c, d) \), so \( h \) is one-to-one.

Therefore, by Corollary 2, \( A \times B \) is countable. \( \bullet \)

Note: If \( A, B, C \) are countable then \( A \times B \) is countable so, by Theorem 4, \( (A \times B) \times C \) is also countable. But \( (A \times B) \times C \cong A \times B \times C \), so \( A \times B \times C \) is countable. Continuing in this way, we can prove by induction that a product of finitely many countable sets is countable.)

**Theorem 5** Suppose, for each \( n \in \mathbb{N} \), that \( A_n \) is countable. Then \( \bigcup_{n=1}^{\infty} A_n \) is countable. 
(The union of a countable collection \( \{A_1, ..., A_n, ...\} \) of countable sets is countable.)

**Proof** To show \( \bigcup_{n=1}^{\infty} A_n \) is countable, it is sufficient, by to produce a one-to-one map \( f: \bigcup_{n=1}^{\infty} A_n \to \mathbb{N} \) (by Corollary 2).

1) We begin by showing that we can find pairwise disjoint countable sets \( B_n \) such that \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \). (Then we will just need to show that \( \bigcup_{n=1}^{\infty} B_n \) is countable.)

Define \( B_1 = A_1 \), 
\( B_2 = A_2 - A_1 \), 
\( B_3 = A_3 - (A_1 \cup A_2) \), 
\[ \vdots \]
\( B_n = A_n - (A_1 \cup ... \cup A_{n-1}) \)
\[ \vdots \]

To see that the \( B_n \)'s are pairwise disjoint:

Consider \( B_m \) and \( B_n \) where, say, \( m < n \). If \( s \in B_m \) \( = A_m - (A_1 \cup ... \cup A_{m-1}) \), then \( x \in A_m \). But that implies \( x \notin B_n = A_n - (A_1 \cup ... \cup A_m \cup ... \cup A_{n-1}) \). So \( B_m \cap B_n = \emptyset \).

To see that \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \):

Since \( B_n \subseteq A_n \), certainly we have \( \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n \).

Conversely, if \( x \in \bigcup_{n=1}^{\infty} A_n \), then \( x \) is in at least one of the \( A_n \)'s. Pick the smallest \( n_0 \) such that \( x \in A_{n_0} \). Then \( x \in B_{n_0} \) because \( x \) is not in \( A_1 \cup ... \cup A_{n_0-1} \), the set “discarded” in the definition of \( B_{n_0} \). Therefore \( x \in \bigcup_{n=1}^{\infty} B_n \), so \( \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n \).

2) Since each \( B_n \) is a subset of a countable set \( A_n \), Corollary 3 tells us that each \( B_n \) is countable. We want to show that \( \bigcup_{n=1}^{\infty} B_n \) is countable.

There is a one-to-one function \( f_n : B_n \to \mathbb{N} \), for each \( n \) (since \( B_n \) is countable).
List the prime numbers as $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, ..., $p_n$, ...

Define $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ as follows: if $x \in B_n$, then $f(x) = p_n^{f_n(x)} \in \mathbb{N}$

For example, if $x \in B_3$, then $f_3 : B_3 \to \mathbb{N}$ and $f_3(x)$ is some natural number. Suppose, to illustrate, that $f_3(x) = 64$. Then $f(x) = p_3^{f_3(x)} = p_3^{64} = 5^{64}$.

Notice that for each $x \in \bigcup_{n=1}^{\infty} B_n$, there is exactly one $n$ for which $x \in B_n$ because the $B_n$'s are pairwise disjoint. This means there's no ambiguity in how $f$ is being defined.

We claim that $f$ is one-to-one. Suppose $x, y \in \bigcup_{n=1}^{\infty} B_n$ and $x \neq y$.

i) if $x \in B_n$ and $y \in B_m$, where $m \neq n$, then

$$f(x) = p_n^{f_n(x)} \neq p_m^{f_m(y)} = f(y) \text{ because } p_n \neq p_m$$

Fundamental Theorem of Arithmetic: $p_n^{f_n(x)}$ can't be factored as a product of $p_m$'s.

ii) if $x$ and $y$ are both in the same $B_n$, then $f_n(x) \neq f_n(y)$ because the function $f_n : B_n \to \mathbb{N}$ is one-to-one. Therefore

$$f(x) = p_n^{f_n(x)} \neq p_n^{f_n(y)} = f(y)$$

Fundamental Theorem of Arithmetic again: $p_n^{f_n(x)}$ and $p_n^{f_n(y)}$ have different numbers of $p_n$'s in their prime factorizations, so they can't be equal.

Therefore $h$ is one-to-one.

Therefore $\bigcup_{n=1}^{\infty} B_n$ is countable. •

**Corollary 6** A union of a finite number of countable sets is countable. (In particular, the union of two countable sets is countable.)

(This corollary is just a minor “fussy” step from Theorem 5. The way Theorem 5 is stated, it applies to an infinite collection of countable sets $A_1, ..., A_n, ...$ If we have only finitely many, we artificially create the others using $\emptyset$.)

**Proof** Suppose $A_1, A_2, ..., A_k$ are countable sets. In order to apply Theorem 5, we define $A_{k+1} = A_{k+2} = A_{k+3} + ... = \emptyset$.

Then $A_1 \cup ... \cup A_k = A_1 \cup ... \cup A_k \cup \emptyset \cup \emptyset \cup ... = \bigcup_{n=1}^{\infty} A_n$, which is countable. •
Examples

i) We proved earlier that the set $\mathbb{Q}^+$ of positive rationals is countable. The set $\mathbb{Q}^-$ of negative rationals is countable since $\mathbb{Q}^- \simeq \mathbb{Q}^+$ (using, for example, the function $f(x) = -x$). The finite set $\{0\}$ is countable.

Therefore $\mathbb{Q} = \mathbb{Q}^- \cup \{0\} \cup \mathbb{Q}^+$ is countable, by Corollary 6.

ii) If $B \subseteq A$, and $B$ is uncountable, then $A$ must be uncountable. (If $A$ were countable, then Corollary 3 would say that $B$ must be countable.)

For example, $\mathbb{R}$ is uncountable because $(0, 1) \subseteq \mathbb{R}$ (in fact, we already knew $\mathbb{R}$ is uncountable – we proved $\mathbb{R} \simeq (0, 1)$).

$\mathbb{R} \times \{0\} \times \{0\}$ is uncountable (it's clearly equivalent to $\mathbb{R}$), and $\mathbb{R} \times \{0\} \times \{0\}$ $\subseteq \mathbb{R}^3$. Therefore $\mathbb{R}^3$ is uncountable.

A similar argument shows that $\mathbb{R}^n$ is uncountable for any integer $n > 1$.

iii) Let $\mathbb{P}$ be the set of irrational numbers. If $\mathbb{P}$ were countable, then $\mathbb{R} = \mathbb{P} \cup \mathbb{Q}$ would be countable (by Corollary 6). Therefore $\mathbb{P}$ is uncountable – so there are “more” irrational numbers than there are rational numbers !!

iv) A harder example: suppose $A = \{a_1, a_2, \ldots, a_n, \ldots\} \subseteq \mathbb{R}$ and let $\epsilon$ be an arbitrary positive number. Let $I_n$ denote the open interval centered at $a_n$ with length $\frac{\epsilon}{2^n n!}$, that is $I_n = (a_n - \frac{\epsilon}{2^n n!}, a_n + \frac{\epsilon}{2^n n!})$. Then $A \subseteq \bigcup_{n=1}^{\infty} I_n$ (because each $a_n$ is in the corresponding $I_n$); but the total length of all the intervals $I_n$ is $\sum_{n=1}^{\infty} \frac{\epsilon}{2^n n!} < \epsilon$.

This says, informally, that any countable subset of $\mathbb{R}$ can be “covered” by a countable collection of open intervals whose total length is $< \epsilon$, where the positive number $\epsilon$ can be chosen as small as you like.

So – for a dramatic and counterintuitive example – you can cover the set $\mathbb{Q}$ of all rational numbers with a countable collection of open intervals whose total length is $< 10^{-10}$ !!