Mathematical Induction

Consider the statement

"if n is even, then $4|n^2$"

As it stands, this statement is neither true nor false: n is a variable and whether the statement is true or false depends on what value of n, from what universe, we're talking about. However,

$(\forall n \in \mathbb{N})$ (if n is even, then $4|n^2$)

is a (true) proposition. It asserts that a certain statement is true for every n in the universe $\mathbb{N}$.

The Principle of Mathematical Induction (PMI) is a method for proving statements of the form $(\forall n \in \mathbb{N})\, P(n)$.

Note: Outside of mathematics, the word "induction" is sometimes used differently. There, it usually refers to the process of making empirical observations and then generalizing from them to a conclusion: for example, we observe the sun coming up in the east thousands of times and conclude that "the sun always rises in the east." This sort of "induction" is an important part of the scientific method. Of course, it raises a philosophical problem:

"How can we ever justify drawing a universal conclusion from particular observations – even if there are thousands of observations?" And if we can't, then why is the scientific method as effective as it is?

Outside of mathematics, induction is usually viewed as "opposite" to deduction. Deduction is a style of argument that starts with certain premises (assumptions) and logically proceeds to a conclusion without reference to any empirical observations. In a correct deductive argument, the conclusion must be true if the premises are true.

For example, a nonmathematician might observe, correctly, that the statement

"if n is a natural number, then $n^2 - n + 41$ is prime"

is true for $n = 1, 2, 3, \ldots, 40$ and then conclude that the statement is true for all natural numbers n. This would be an example of a conclusion drawn by "induction" in the everyday use of the word. But this conclusion is incorrect: for $n = 41$, $n^2 - n + 41 = 41^2 - 41 + 41 = 41^2$ is not prime.

"Mathematical induction" is something totally different. It refers to a kind of deductive argument, a logically rigorous method of proof. It works because of how the natural numbers are constructed from set theory; as we shall see later, PMI is "built into $\mathbb{N}$."

First we will state PMI and a variation (called PCI, the Principle of Complete Induction) in the form in which they most often used.

Suppose $P(n)$ is a statement about a natural number n.

Principle of Mathematical Induction (PMI)

If $\begin{cases} \text{i) } P(1) \text{ is true, and} \\ \text{ii) when } P(k) \text{ is true for some particular } k \in \mathbb{N}, \\ \quad \text{then } P(k+1) \text{ must also be true} \end{cases}$ then $\forall n \in \mathbb{N}\ P(n)$ is true

Principle of Complete Induction (PCI)

If $\begin{cases} \text{i) } P(1) \text{ is true, and} \\ \text{ii) when } k > 1 \text{ and } P(n) \text{ is true for every } n < k, \\ \quad \text{then } P(k) \text{ must also be true} \end{cases}$ then $\forall n \in \mathbb{N}\ P(n)$ is true

Both of these principles seem intuitively clear (*see the discussion in class*). We will see why they are actually equivalent. For the moment, let's just practice using them.

Example 1 Let $P(n)$ be the statement : $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ or more informally $1 + \ldots + n = \frac{n(n+1)}{2}$

Prove that $(\forall n \in \mathbb{N})\ P(n)$ is true.

Proof i) *Step i) is called the base case for the induction – the "starting point."*

$P(1)$ is the statement $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$, which is true.

ii) Assume $P(k)$ is true for some $k \in \mathbb{N}$.

This assumption is called the induction hypothesis or induction assumption. Note: we are not assuming here that $P(k)$ is true for every k in $\mathbb{N}$ – that would be assuming the very thing we're trying to prove! Rather, we are supposing that k is a particular value of n for which $P(n)$ is true (such as, for example, $k = 1$).

Assuming this, we need to show that $P(k+1)$ must also be true.

A good idea: write out $P(k+1)$ to be clear about what it says. $P(k+1)$ reads:
$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$ *or, more informally,* $1 + \ldots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

To do this:

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)+2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$\uparrow$ *by the induction hypothesis*

which says that $P(k+1)$ is true. By PMI, $\forall n \in \mathbb{N}\ P(n)$ is true – that is, $P(n)$ is true for all $n \in \mathbb{N}$. •

Example 2 (PCI) Prove that every natural number $n > 1$ is either a prime or a product of primes. *If we count a prime number as being a product of primes with "just one factor," then we could say that "every natural number bigger than* 1 *can be factored into primes."*
This is one part of the Fundamental Theorem of Arithmetic; the other part (which we will not prove until later) says that the factorization is unique except for the order of the factors.

More formally, we are supposed to prove: $\forall n \in \mathbb{N}\ \ P(n)$, where $P(n)$ is the (equivalent) statement:

$(n = 1) \vee (n$ is prime$) \vee (n$ can be factored into primes$)$

Proof i) $P(1)$ is true since $1 = 1$.

ii) Suppose $k > 1$ and assume $P(n)$ is true for all $n < k$. (*This is the "induction hypothesis."*) We must show $P(k)$ is true.

If k is prime, then $P(k)$ is true. Otherwise, since $k > 1$, we can factor $k = pq$ where $p, q \in \mathbb{N}$ and $p, q < k$. The factor p is either prime or (by the induction hypothesis) p factors into primes; and the same is true for q. Therefore k factors into primes.

By PCI, we conclude that $\forall n \in \mathbb{N}\ P(n)$ is true. •

Notice the <u>similarities and differences</u> between using PMI and PCI as illustrated in Examples 1 and 2.

With both methods, we need to start by verifying a base case: in Example 1-2, that $P(1)$ is true.

With both methods, we need to prove that $P(n)$ is true for a certain value of n – under a certain assumption. This is the "induction step."

a) With MI (Example 1), we need to show, <u>assuming</u> that $P(k)$ is true for some value k, that $P(k+1)$ is also true : the induction <u>hypothesis</u> <u>only</u> <u>involves</u> a single natural number k, the "immediate predecessor" of $k+1$.

To illustrate: With PMI, the induction step shows, for example, that if $P(3)$ is true, then $P(4)$ must also be true.

b) With PCI (Example 2), we need to show, <u>assuming</u> that $P(n)$ is true for <u>all</u> values of n preceding some k, that $P(k)$ is also true. The induction hypothesis involves <u>all</u> the natural numbers preceding k.

To illustrate: With PCI, then induction step shows, for example, that if $P(1)$ <u>and</u> $P(2)$ <u>and</u> $P(3)$ are true, then $P(4)$ must be true.

The shift from using $k+1$ in the induction step (in PMI) to using k in the induction step (in PCI) is a just a minor notational shift that is convenient and traditional: it's not really a significant "difference" in the methods. The <u>main difference</u> is that PCI seems to give us much more to work with than PMI: with PCI, we assume that <u>all</u> of $P(1), \ldots, P(k-1)$ are true and try to use this information prove $P(k)$. PCI seems to give us many more assumptions to use in proving $P(k)$ is true.

In light of this observation, it may be surprising that PMI and PCI turn out to be logically equivalent: either both are true statements about $\mathbb{N}$ or both are false statements about $\mathbb{N}$. If we assume PCI, we can prove PMI is also true, and vice-versa. (*We will show this soon.*) However, this does not mean that they are equally useful in a given situation.

> In Example 1, all we need to prove $P(k+1)$ is $P(k)$; any additional assumptions about $P(k-1), .., P(1)$ are completely unnecessary.
>
> In Example 2, it's hard to see how we could prove that k factors into primes if the induction assumption were only about the single number preceding k $-$ that is, if the induction assumption were merely that $k-1$ factors into primes. In the proof in Example 2, we need to know, somehow, that p and q are products of primes and that's what the induction hypothesis using PCI gives us!

Here's a more vivid illustration of PMI and PCI that might fix them in your memory:

> **PMI states that i) if 1 is red and ii) if a natural number must be red whenever its immediate predecessor is red, then all natural numbers must be red.**
>
> **PCI states that i) if 1 is red and ii) if a natural number must be red whenever all its predecessors are red, then all natural numbers must be red.**

There is another useful property of $\mathbb{N}$, called the well-ordering principle, that can sometimes be used to prove a statement of the form $\forall n \in \mathbb{N}\ P(n)$. It is stated in terms of subsets of $\mathbb{N}$.

Well-Ordering Principle (WOP) If $A \subseteq \mathbb{N}$ and $A \neq \emptyset$, then A contains a smallest element.

> *Intuitively, this seems clear: if there were a nonempty subset A of $\mathbb{N}$ with no smallest element, we could choose a number $a_1 \in A$; since A has no smallest element, there would be an $a_2 \in A$ where $a_2 < a_1$; since a_2 is not the smallest element in A, there would be a stiller smaller number $a_3 \in A$. Continuing in this way we could write down an infinite decreasing sequence of natural numbers: $a_1 > a_2 > a_3 > \ldots > a_n > \ldots$ and this is intuitively impossible. Try it!*
>
> > *Suppose, say, $a_1 = 103217$; what could a_2 be? a_3 ? Can you find an "infinite descending sequence" $103217 > a_2 > a_3 > \ldots > a_n > \ldots$?*
>
> *Notice that in this intuitive argument, we need that $A \neq \emptyset$: that assumption is how we get a_1 in A to start with. In fact, if $A = \emptyset$, then A doesn't contain a smallest element $-$ because A has no elements at all.*

Can we justify rigorously why WOP a true statement about $\mathbb{N}$? It turns out (another surprise?) that all three of WOP, PMI and PCI are logically equivalent. As we will see later, PMI is "built-into" the system $\mathbb{N}$ at a very fundamental level, when $\mathbb{N}$ is constructed from set theory and therefore PCI and WOP are automatically "built-in" too. All three are very fundamental facts about $\mathbb{N}$.

Because PMI, PCI and WOP are logically equivalent, any proof that can be done using one them could also be done (at least in principle) using the other. Sometimes one is more convenient to use than the other, and sometimes choosing between them is just a matter of taste.

The following example gives a proof of the result in Example 1 using WOP instead of PMI. Notice the difference in the approach; but equally important, notice the similarities in the algebra that comes up. You can decide whether you prefer this argument to the one using PMI in Example 1.

Example 3 Use WOP to prove: $(\forall n \in \mathbb{N})\, P(n)$, where $P(n)$ is the statement: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Proof Let $A = \{n \in \mathbb{N} :\ P(n)$ is false$\}$.

To complete the proof, we want to show that $A = \emptyset$. *We argue by contradiction.*

Assume $A \neq \emptyset$. Then, by WOP, there must be a smallest k in A: that is,

$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is false (*) and

$P(n)$ is true for natural numbers n smaller than k.

Since $\sum_{i=1}^{1} i = \frac{1(1+1)}{2}$ is true, we know $k \neq 1$, so $k > 1$. Therefore $n = k - 1$ is still a natural number and smaller than k, so $P(k-1)$ must be true:

$$\sum_{i=1}^{k-1} i = \frac{(k-1)((k-1)+1)}{2} = \frac{(k-1)(k)}{2} \qquad \text{is true.} \quad (**)$$

Then adding k to both sides of (**) gives that $\sum_{i=1}^{k-1} i + k = \frac{(k-1)(k)}{2} + k$ is true.

But this equation (*do the algebra!*) simplifies to $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ – which contradicts $(*)$.

Since the assumption that $A \neq \emptyset$ leads to a contradiction, it must be that $A = \emptyset$; in other words,

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$. •

Mathematical induction (in any of the equivalent forms PMI, PCI, WOP) is not just used to prove equations. Example 2, in fact, uses PCI to prove part of the Fundamental Theorem of Arithmetic. Examples 4 and 5 illustrate using induction to prove an inequality and to prove a result in calculus.

Example 4 Prove that $e^n > 1 + n$ for all natural numbers n.
More formally, we want to prove $(\forall n \in \mathbb{N})\, P(n)$, where $P(n)$ is the statement: $e^n > 1 + n$

Proof i) $P(1)$ states that $e > 2$, which is known to be true. (*From calculus:* $e =$ 2.718281828459045...)

ii) (*Induction step*) Assume that $P(k)$ is true for some particular $k \in \mathbb{N}$: for this k, $e^k > 1 + k$.

We need to prove that $P(k+1)$ *must be true, that is* : $e^{k+1} > 1 + (k+1) = 2 + k$

$$e^{k+1} = e^k \cdot e \;\; > (1+k) \cdot e \;\; > (1+k) \cdot 2 = 2 + 2k > 2 + k$$

$\uparrow$ *by the induction hypothesis* $\quad$ $\uparrow$ *because* $e > 2$ $\quad$ $\uparrow$ *because* $k \in \mathbb{N},\ 2k > k$.

So $P(k+1)$ is true.

By PMI, $(\forall n \in \mathbb{N})\, P(n)$ is true. •

Example 5 Prove that if n is a natural number, then $\frac{d}{dx} x^n = n x^{n-1}$. (*You many assume the product rule for derivatives.*)

Proof Let $P(n)$ be the statement: $\frac{d}{dx} x^n = n x^{n-1}$

i) $P(1)$ is the statement that $\frac{d}{dx} x = 1 \cdot x^{1-1} = 1 \cdot x^0 = 1$

This is true from calculus, but we assume that you could prove it using the definition of derivative: for $f(x) = x$, $\frac{d}{dx} x = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim\limits_{h \to 0} \frac{(x+h) - x}{h} = \lim\limits_{h \to 0} \frac{h}{h} = \lim\limits_{h \to 0} 1 = 1.$

ii) Assume that $\frac{d}{dx} x^k = k x^{k-1}$ for some particular $k \in \mathbb{N}$ (*the induction hypothesis*).

$$\frac{d}{dx} x^{k+1} = \frac{d}{dx}(x^k \cdot x) = \left(\frac{d}{dx} x^k\right) \cdot x + x^k \left(\frac{d}{dx} x\right) = (k x^{k-1}) x + x^k (1) = k x^k + x^k = (k+1) x^k$$

$\uparrow$ *the product rule* $\quad$ $\uparrow$ *by the induction hypothesis* <u>*and*</u> *because* $P(1)$ *is true*

$= (k+1) x^{(k+1)-1}$, so $P(k+1)$ is true.

By PMI, it follows that $(\forall n \in \mathbb{N})\ P(n)$ is true. •

Rephrasings of PMI and PCI

WOP is stated in terms of sets: every nonempty subset of $\mathbb{N}$ contains a smallest element. To prove that the three statements are equivalent, we will reformulate PMI and PCI in the language of set theory. Then each of PMI, PCI, WOP will be a statements about certain subsets of $\mathbb{N}$

Suppose $P(n)$ is a statement about a natural number n.

Principle of Mathematical Induction (PMI)

If $\begin{cases} \text{i) } P(1) \text{ is true, and} \\ \text{ii) when } P(k) \text{ is true for some particular } k \in \mathbb{N}, \\ \quad \text{then } P(k+1) \text{ must also be true} \end{cases}$ then $\forall n \in \mathbb{N}\ P(n)$ is true

PMI rephrased as a statement about subsets of $\mathbb{N}$

Suppose $S \subseteq \mathbb{N}$.

If $\begin{cases} \text{i) } 1 \in S, \text{ and} \\ \text{ii) } \forall k\ (\textit{if } k \in S, \textit{ then } k+1 \in S) \end{cases}$ then $S = \mathbb{N}$

When we actually use PMI to prove a statement $\forall n \in \mathbb{N}\ P(n)$, *we are setting* $S = \{n \in \mathbb{N} : P(n) \textit{ is true}\}$. *Then*

i) checking that "$P(1)$ is true" is the same as checking that "$1 \in S$," and
ii) checking that "if $P(k)$ is true for some k, then $P(k+1)$ must also be true"
is the same as checking that "if $k \in S$, then $k+1 \in S$"

Principle of Complete Induction (PCI)

If $\begin{cases} \text{i) } P(1) \text{ is true, and} \\ \text{ii) when } k > 1 \text{ and } P(n) \text{ is true for every } n < k, \\ \quad \text{then } P(k) \text{ must also be true} \end{cases}$ then $\forall n \in \mathbb{N}\ P(n)$ is true

PCI rephrased as a statement about subsets of $\mathbb{N}$

Suppose $S \subseteq \mathbb{N}$.

If $\begin{cases} \text{i) } 1 \in S, \text{ and} \\ \text{ii) when } k > 1 \text{ and } \{n : n < k\} \subseteq S, \text{ then } k \in S \end{cases}$, then $S = \mathbb{N}$

When we actually use PCI to prove a statement $\forall n \in \mathbb{N}\ P(n)$, *we are setting* $S = \{n \in \mathbb{N} : P(n) \textit{ is true}\}$. *Then*

i) checking that "$P(1)$ is true" is the same as checking that "$1 \in S$," and
ii) checking that
"when $k > 1$ and $P(n)$ is true for every $n < k$, then $P(k)$ must also be true"
is the same as checking that
"when $k > 1$ and $n \in S$ for every $n < k$, then $k \in S$.

(*See comments on the next page*)

I stated PCI as I did on the preceding page (with two parts, i) and ii), because it's the set theoretic version of what we usually do in practice.

However, it's possible to state the set theoretic version of PCI more efficiently *(and this is what the textbook does)* without mentioning i) at all. Simply drop "$k > 1$" from ii) and rewrite PCI as follows:

Suppose $S \subseteq \mathbb{N}$

(*) If $\forall k \in \mathbb{N}\ \ (\{n \in \mathbb{N} : n < k\} \subseteq S \Rightarrow k \in S\})$, then $S = \mathbb{N}$

According to (*), to prove that $S = \mathbb{N}$ we should look at all k in $\mathbb{N}$ (not just $k > 1$, as in Part ii) on the preceding page) and argue that $\{n \in \mathbb{N} : n < k\} \subseteq S \Rightarrow k \in S$.

For $k = 1$, what do we do to show that $\{n \in \mathbb{N} : n < k\} \subseteq S \Rightarrow k \in S$ is true? Notice that $\{n \in \mathbb{N} : n < 1\} \subseteq S$ is true automatically, because $\{n \in \mathbb{N} : n < k\} = \emptyset$.
So, the conditional statement $\{n \in \mathbb{N} : n < 1\} \subseteq S \Rightarrow 1 \in S$ is true iff $1 \in S$, that is, iff $P(1)$ is true. And verifying the $P(1)$ is true is precisely what Sept i) says on the previous page.

So, if we are using PCI and check that

(*) $\forall k \in \mathbb{N}\ \ (\{n \in \mathbb{N} : n < k\} \subseteq S \Rightarrow k \in S\})$

we have automatically completed Step i)

So why did I break off Step i) as a separate step? It wasn't necessary to do that. But when $k = 1$, there are no n's $< k$, so when we try to show $P(1)$ is true, there aren't any "previous values of n" for which $P(n)$ is assumed to be true – that is, $P(1)$ has to be established "from scratch" without any "induction assumptions" to work with and therefore it's usually done in a different way than when $k > 1$. Therefore, in practice, we might as well state the case $k = 1$ separately. It's probably a little clearer that way, even if not quite so efficient as (*).

As a reminder, we restate WOP here.

III. WOP (Well-Ordering Principle) Suppose $A \subseteq \mathbb{N}$.

If $A \neq \emptyset$, then A contains a smallest element.

The Equivalence of PMI, PCI and WOP

Theorem PMI, PCI, and WOP are equivalent.

Perhaps you feel that WOP is easier to believe than the other two. But (to repeat) PMI, PCI and WOP are equivalent statements: either all three are true statements in $\mathbb{N}$, or none of them is true in $\mathbb{N}$.

Since $\mathbb{N}$ is constructed from set theory in a way that makes PMI true (as we shall see), all three statements are true in $\mathbb{N}$.

Proof We will show that the three statements are equivalent by giving three separate arguments to show that:

i) PMI $\Rightarrow$ PCI   ii) PCI $\Rightarrow$ WOP   iii) WOP $\Rightarrow$ PMI.

i) Prove PMI $\Rightarrow$ PCI

Assume that PMI is true. Let $S \subseteq \mathbb{N}$.

To prove PCI is true:

We assume that $\forall k \in \mathbb{N}\ (\{n \in \mathbb{N} : n < k\} \subseteq S\ \Rightarrow k \in S\})$ (*)
and we need to show that $S = \mathbb{N}$.

Strategy: We will show (using PMI) that for every n, $\{1, 2, ..., n\} \subseteq S$. *If that is true, then* $n \in S$ *for every* n, *which tells us that* $\mathbb{N} \subseteq S$. *Since we already know* $S \subseteq \mathbb{N}$, *we will conclude that* $S = \mathbb{N}$.

a) Since $\{n : n < 1\} = \emptyset \subseteq S$, (*) gives that $1 \in S$. So $\{1\} \subseteq S$.

b) Assume that for some particular value k,, we have $\{1, 2, ..., k\} \subseteq S -$ that is, assume $\{n : n < k+1\} \subseteq S$. By (*), we conclude that $k+1 \in S$. Therefore $\{1, 2, ..., k, k+1\} \subseteq S$.

Summarizing: $1 \in S$ *and* (*if* $\{1, 2, ..., k\} \subseteq S$, *then* $\{1, 2, ..., k+1\} \subseteq S$)

Using PMI, we conclude that for $\forall n\ \{1, 2, ..., n\} \subseteq S$. Therefore $\mathbb{N} \subseteq S$, so (*as outlined in the strategy*), $S = \mathbb{N}$. •

ii) Prove PCI $\Rightarrow$ WOP

Assume PCI is true. Suppose $A \subseteq \mathbb{N}$.

We want to prove WOP: if $A \neq \emptyset$, then A contains a smallest element.

Strategy: We will prove the contrapositive of WOP instead.

Assume A contains no smallest element; we will prove that $A = \emptyset$.
Let $S = \mathbb{N} - A$.

i) Since A has no smallest element, $1 \notin A$, so $1 \in S$. Since $1 \in S$ is true, the conditional statement $(\{k : k < 1\}) \Rightarrow (1 \in S)$ is true.

ii) Suppose $k > 1$ and that $\{n : n < k\} = \{1, 2, ..., k-1\} \subseteq S$. By definition of S, this means that $1, 2, ..., k-1$ are not in A. Therefore $k \notin A$ (if $k \in A$, k would be the smallest element in A), so $k \in S$.

Because of i) and ii), PCI tells us that $S = \mathbb{N}$, and therefore $A = \emptyset$. •

iii) Prove WOP $\Rightarrow$ PMI

Assume that WOP is true. Suppose $S \subseteq \mathbb{N}$. We want to prove PMI:

If $1 \in S$ and $\forall k\,(k \in S \Rightarrow k+1 \in S)$, then $S = \mathbb{N}$.

Strategy: PMI, as stated above, has the form $P \wedge Q \Rightarrow R$,
which is equivalent to $P \wedge \sim R \Rightarrow \sim Q$. So PMI is equivalent to

If $1 \in S$ and $S \neq \mathbb{N}$, then $\sim (\forall k)(k \in S \Rightarrow k+1 \in S)$

and this is equivalent to

If $1 \in S$ and $S \neq \mathbb{N}$, then $(\exists k)\,(\,k \in S \;\wedge\; k+1 \notin S)$ (#)

Assuming WOP, we will prove (#).

Suppose $1 \in S$ and $S \neq \mathbb{N}$. Then $\mathbb{N} - S \neq \emptyset$, so by WOP there is a smallest natural number in $\mathbb{N} - S$: call it n.

Since $n \notin S$, we know $n > 1$. Therefore $k = n-1$ is still natural number, and since n is the smallest natural number not in S, we must have $n - 1 \in S$. Let $k = n-1$. Then $k \in S$ but $k + 1 = n \notin S$. This proves (*) (*see the strategy*). •

There are minor generalizations of PMI and PCI that are easy to use. Essentially they say that the "base case" for the induction can be any integer k_0; we don't have to start at 1.
We put this at the end of these notes because, in fact, we could easily do without them (*see the final two examples*). For example, here is the Generalized PMI; there is a similar generalization for PCI, which you can easily write down.

Notice that here $\mathbb{Z}$ is used in the hypothesis instead of $\mathbb{N}$. The only reason for that is to allow the base case ("starting point") for the induction to possibly be a negative integer n_0. The induction could start, say, at $n = -3$. But in that case, the conclusion is NOT $S = \mathbb{Z}$; just that $S = \{n \in \mathbb{Z} : n \geq -3\}$ = "all integers $\geq$ the base case."

> Generalized Principle of Mathematical Induction (GPMI) Suppose $S \subseteq \mathbb{Z}$ and that $k_0 \in \mathbb{Z}$. If
>
> i) $k_0 \in S$, and
>
> ii) for any $k \geq k_0$, $(k \in S \Rightarrow k+1 \in S)$
>
> then $S = \{n \in \mathbb{Z} : n \geq k_0\}$

To use GPMI: If we want to prove that $P(n)$ is true for all $n \geq 7$ (*or all* $n \geq -3$) we would

i) Show that $P(7)$ is true (*or that* $P(-3)$ *is true*). Then

ii) Assume that $P(k)$ is true for some particular $k \geq 7$ (*or some* $k \geq -3$)
and argue that $P(k+1)$ must also be true.

Example Prove that $2^n > n^2 + 10n + 3$ for all natural numbers $n \geq 7$
We use GPMI with the starting point (base case) $k_0 = 7$.

i) $128 = 2^7 > 7^2 + 10(7) + 3 = 122$, so $P(7)$ is true
ii) Suppose k is some particular natural number with $k \geq 7$ and that $P(k)$ is true.

> *We need to show that* $P(k+1)$ *is true:* $2^{k+1} > (k+1)^2 + 10(k+1) + 3$

$$2^{k+1} = 2(2^k) \underset{\substack{\uparrow \\ \textit{induction hypothesis}}}{>} 2(k^2 + 10k + 3) = 2k^2 + 20k + 6 = (k^2 + 2k + 1) + k^2 + 18k + 5$$

$$= (k+1)^2 + k^2 + 18k + 5 = (k+1)^2 + 10(k+1) + k^2 + 8k - 5$$

$$= [(k+1)^2 + 10(k+1) + 3] + (k^2 + 8k - 8)$$

$\uparrow$
(*This part is* $P(k+1)$; *I deliberately rearranged terms, above, to make* $P(k+1)$ *appear*)

But $k^2 + 8k - 8 = (k+4)^2 - 24 \geq 0$ because $k \geq 7$. Therefore
$2^{k+1} > (k+1)^2 + 10(k+1) + 3$, so $P(k+1)$ is true.

By GPMI, $(\forall n \in \mathbb{N})\ (n \geq 7 \Rightarrow 2^n > n^2 + 10n + 3)$ is true. •

Using Generalized GPMI is just a matter of convenience; we could avoid it altogether just by making a substitution that translates the variable, followed by using PMI. For example, instead of using GPMI to prove

$$(*) \qquad (\forall n \in \mathbb{N})\ \ (n \geq 7 \Rightarrow 2^n > n^2 + 10n + 3)$$

we could "translate"; another way of saying the same thing is

$$(\forall n \in \mathbb{N})\ \ (n \geq 1 \Rightarrow 2^{n+6} > (n+6)^2 + 10(n+6) + 3)$$

Since every natural number n automatically satisfies $n \geq 1$, the last statement is equivalent to

$$(**) \qquad (\forall n \in \mathbb{N})\ \ 2^{n+6} > (n+6)^2 + 10(n+6) + 3,$$

which we can prove using PMI.

(*Be sure you're convinced that* $(*)$ *and* $(**)$ *are equivalent.*)