Example

Prove: there is a rational number between any two positive real numbers.

More formally stated. Suppose that x and y real numbers. Prove that

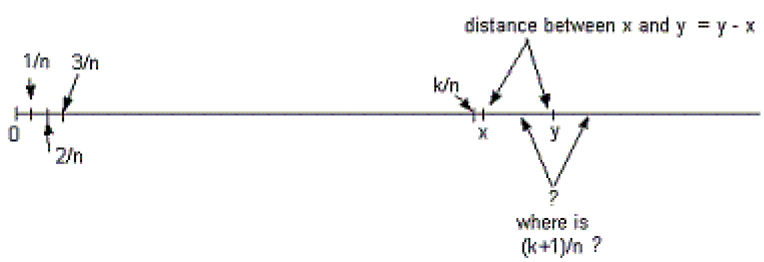
$$(\forall x)(\forall y)\ (x > 0\ \wedge y > 0\ \wedge x < y) \Rightarrow (\exists z)\,(z \text{ is rational } \wedge x < z < y)$$

Proof Let x and y be real numbers. Assume x and y are positive and that $x < y$. Since $y - x > 0$, we can pick a natural number n large enough to make $\frac{1}{n} < y - x$. Look at the numbers $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, ..., \frac{k}{n}$ and pick the <u>largest possible</u> natural number k for which $\frac{k}{n} \leq x$ (*see the figure below*).

Then $x < \frac{k+1}{n}$ (because of how we chose k). In addition, $\frac{k+1}{n} < y$. (*Here, we have a short proof by contradiction nestled inside of the main proof. An additional indent may help guide the reader's eye.*)

> Assume $y \leq \frac{k+1}{n}$. Then $\frac{1}{n} = \frac{k+1}{n} - \frac{k}{n} \geq y - x$, which is false (because of how we chose n.)

Thus $z = \frac{k+1}{n}$ is a rational number which satisfies $x < z < y$. ●



Note: The figure is not an "official" part of the proof. But the inclusion of a figure certainly can help the reader understand what's going on. Just be sure that the actual proof can actually stand on its own (through its algebra, etc.) without actually <u>*using*</u> *the picture.*

(OVER)

The name “corollary” is often used, instead of theorem, for a result that is proven as a relatively easy consequence of a theorem that's already been proven. (The etymology is: *corollarium “a deduction, consequence,” from Latin corollarium, originally “money paid for a garland,” hence “gift, gratuity, something extra” from corolla “small garland,” dim. of corona “crown.”*)

As an exercise, prove the following corollary to the theorem given above.

Corollary There is a rational number between any two real numbers.

Hint: Consider cases. If x and y are both positive, we have already proven the result. What happens if x is negative and y is positive? What if both are negative? In the latter two cases, you could try to mimic the proof of theorem, but try to do it in an easier way. One of the two cases has an almost trivial proof; deduce the result in the remaining case by thinking about how to use the theorem we already proved.