Length of $y = \ln x^3 = 3\ln x$, for $1 \le x \le 4$

$L = \int_1^4 \sqrt{1 + (\frac{dy}{dx})^2}\, dx \; = \int_1^4 \sqrt{1 + \frac{9}{x^2}}\; dx.$

We'd like $\sqrt{1+\tan^2\theta}$ as the integrand, so we want $\frac{9}{x^2} = \tan^2\theta$. Therefore we substitute

$x = \frac{3}{\tan\theta},\ dx = \frac{-3\sec^2\theta}{\tan^2\theta}\mathrm{d}\theta.$

This gives $\int \sqrt{1+\frac{9}{x^2}}\; dx = \int \sqrt{1+\tan^2\theta}\; \cdot (-\; \frac{3\sec^2\theta}{\tan^2\theta})\mathrm{d}\theta$

$= \; -3\int \frac{\sec^3\theta}{\tan^2\theta} d\theta = \; -3\int \frac{1}{\cos^3\theta} \cdot \frac{cos^2\theta}{\sin^2\theta} d\theta = \; -3\int \frac{1}{\cos\theta} \cdot \frac{1}{\sin^2\theta} d\theta = \; -3\int \sec\theta \csc^2\theta\; d\theta =$

$-3\int \sec\theta\,(1+\cot^2\theta)\; d\theta = \; -3\int \sec\theta + \sec\theta\cot^2\theta\; d\theta = \; -3\int \sec\theta + \frac{1}{\cos\theta} \cdot \frac{\cos^2\theta}{\sin^2\theta}\; d\theta$

$= \; -3\int \sec\theta + \frac{\cos\theta}{\sin^2\theta}\; d\theta = \; -3\int \sec\theta + \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{\sin\theta}\; d\theta$

$= \; -3(\int \sec\theta\; \mathrm{d}\theta + \int \csc\theta\cot\theta\; d\theta$

$= \; -3\; (\,\ln|\sec\theta + \tan\theta) - \csc\theta\,)$ (since $\frac{3}{x} = \tan\theta$ and therefore $\theta = \arctan(\frac{3}{x})$
)

$= \; -3(\ln|\sec(\arctan\frac{3}{x}) + \tan\arctan\frac{3}{x}| - \csc(\arctan\frac{3}{x})\,)$ (draw a pic of triangle with $\theta = \arctan\frac{3}{x}$ to simplify)

$= \; -3(\,\ln|\frac{\sqrt{9+x^2}}{x} \; + \frac{3}{x}| - \frac{\sqrt{9+x^2}}{3}\,) = \; -3\ln|\frac{\sqrt{9+x^2}}{x} \; + \frac{3}{x}| \; + \; \sqrt{9+x^2}$

Evaluting the antiderivative between $a = 1$ and $x = 4$ then gives the correct answer (which I checked by just writing down the original integral and doing a numerical intergration (without using the antiderivative) on a calculator.