Maximum and Minimum of a Continuous Function f on a Closed Interval $[a, b]$

This is a quick review of material you should know. (In the text, it is part of Section 4.2)

Extreme Value Theorem If f is a continuous function defined on a closed interval $[a, b]$, then f must have an absolute maximum value and an absolute minimum value.

In the text, (p. 273), there a figures illustrating that f might not have an absolute maximum or minimum if it isn't continuous or if we are not working with a closed interval (one that includes both endpoints).

If f is a continuous function defined on a closed interval $[a, b]$, how to we find the absolute maximum and minimum values?

Since we know before we start that they must exist, we ask "at what points $x =$? could they possibly occur?" They might occur

i) at one of the endpoints, $x = a$ or $x = b$.

Think of $f(x) = x^2$ on the interval $[-2, 2]$. Its absolute maximum value is 4, happens at $x = -2$ and $x = 2$, the endpoints. (The absolute maximum or minimum value may occur at more than one point in the interval.)

ii) at a point x in $[a, b]$ where $f'(x)$ doesn't exist.

Think of $f(x) = |x|$ on the interval $[-2, 1]$. The absolute maximum value is 2, occurring at the left endpoint $x = -2$. The absolute minimum value is 0, occurring at $x = 0$ where $f'(0)$ doesn't exist (because the curve has a sharp corner).

iii) at a critical point x in (a, b), that is, a point where $f'(x) = 0$.

Think of $f(x) = x^2$ on the interval $[-1, 2]$. The absolute maximum value is 4, occurring at the right endpoint, $x = 2$. The absolute minimum value is 0, occurring at $x = 0$, a point where $f'(0) = 0$.

Therefore to find where the absolute maximum and minimum occur, we make a list of all the "test" points – points of type i), ii), or iii), and then evaluate f at each test point. Since there has to be an absolute maximum value, it must happen at one of the test points; we just compute to see which one (or maybe more than one) produces the largest value for f. (And similarly find the absolute minimum value.)

Example $f(x) = \sin x + \cos x$ on the interval $[0, 2\pi]$.

i) The endpoints 0 and 2π are test points
ii) $f'(x) = \cos x - \sin x$ is defined for every x so there are no test points of type ii)
iii) Setting $\cos x - \sin x = 0$, we get $\tan x = 1$, so $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ are test points.

We then check:

Test points	$x =$	0	2π	$\frac{\pi}{4}$	$\frac{5\pi}{4}$
Value	$f(x) =$	1	1	$\sqrt{2}$	$-\sqrt{2}$

The maximum value of f over the interval $[0, 2\pi]$ is $\sqrt{2}$, occurring at $x = \frac{\pi}{4}$
The minimum value of f over the interval $[0, 2\pi]$ is $-\sqrt{2}$, occurring at $x = \frac{\pi}{4}$