

Error “Control” Formulas

Whenever we approximate $\int_a^b f(x) dx$, we think of

$$\begin{aligned} (*) \int_a^b f(x) dx &= \text{APPROXIMATION} + \text{ERROR, so that} \\ \text{ERROR} &= \int_a^b f(x) dx - \text{APPROXIMATION} \end{aligned}$$

For example, if we're using Simpson's approximation with $n = 10$, we write

$$\text{ERROR} = \int_a^b f(x) dx - S_{10}$$

Issues:

1) Of course we shouldn't expect any simple ways to exactly calculate the ERROR – because if we knew ERROR and knew APPROXIMATION exactly, we could substitute in (*) and then we'd have the exact value of $\int_a^b f(x) dx$. Finding the exact value of ERROR would seem to be as hard as finding the exact value of $\int_a^b f(x) dx$ itself.

2) Also, with complicated functions and approximation methods (like Simpson's Rule), it's not always easy to predict whether the error will be positive (approximation too small) or negative (approximation too big).

In light of 1) and 2) we settle for something that dodges both issues: we only think about $|\text{ERROR}|$ (that is, we'll ignore the sign) and we'll consider formulas that tell us

$$|\text{ERROR}| \leq (\text{****})$$

Such an inequality doesn't give us the value of ERROR, but the inequality says that $|\text{ERROR}|$ is at most (****). That is, the inequality gives us the “worst case scenario.”

Before we actually look at some formulas of that sort, let's see how they would be used to make a statement about the value of $\int_a^b f(x) dx$.

Example Suppose we approximated $\int_a^b f(x) dx$ by using S_{10} , and got $S_{10} = 0.7115$. And suppose that some formula told us that $|\text{ERROR}| \leq 0.0002$. This would mean that

$$|\text{ERROR}| = \left| \int_a^b f(x) dx - S_{10} \right| \leq 0.0002$$

That is (since we don't know the sign of ERROR), S_{10} is “off” by at most ± 0.0002 . Substituting our value for S_{10} and removing the absolute value signs would then give:

$$- 0.0002 \leq \int_a^b f(x) dx - 0.7115 \leq 0.0002 \text{ and then adding } 0.7115 \text{ gives}$$

$$0.7113 \leq \int_a^b f(x) dx \leq 0.7117$$

The “error control” formulas for the trapezoidal, midpoint and Simpson approximations are given below. They are each hard to prove, but we'll examine in class some of the features of the formulas that seem to make sense. (There are similar formulas involving the left and right endpoint approximations, but these approximations are usually so much less accurate than the others that we'll omit them).

We'll refer to ERROR for the trapezoidal, midpoint and Simpson approximation by $ERROR_T$, $ERROR_M$, and $ERROR_S$ or, for short, just E_T , E_M , E_S .

If you approximate $\int_a^b f(x) dx$ by T_n , M_n , or S_n ,

Trapezoidal “error control”	$ E_T \leq \frac{K(b-a)^3}{12n^2}$	where K is a constant <u>you must choose</u> so that $ f''(x) \leq K$ on $[a, b]$.
Midpoint “error control”	$ E_M \leq \frac{K(b-a)^3}{24n^2}$	where K is a constant <u>you must choose</u> so that $ f''(x) \leq K$ on $[a, b]$.
Simpson “error control”	$ E_S \leq \frac{K(b-a)^5}{180n^4}$	where K is a constant <u>you must choose</u> so that $ f^{(4)}(x) \leq K$ on $[a, b]$. (Note: $f^{(4)}(x) = f''''(x)$)

Comments (to be elaborated in the lecture)

- 1) In all cases “how bad” the error is depends on some features of the shape of the graph of $y = f(x)$ – as “measured” by the size of $f''(x)$ or $f''''(x)$
- 2) You might think a bit and choose a particular K for which, say, $|f''(x)| \leq K$ is true. Somehow else might work a bit harder and find a smaller K for which $|f''(x)| \leq K$ is also true. Neither student will draw wrong conclusion, but the second student will draw better conclusions: s/he will realize that the error really can't be quite as bad as the first student thinks it could be.
- 3) In all cases the formulas show that a longer interval $[a, b]$ tends to increase the possible error, and that a larger n tends to decrease the possible error.