ISOPARAMETRIC HYPERSURFACES WITH FOUR PRINCIPAL CURVATURES

THOMAS E. CECIL, QUO-SHIN CHI, AND GARY R. JENSEN

Abstract. Let $M$ be an isoparametric hypersurface in the sphere $S^n$ with four distinct principal curvatures. Münzner showed that the four principal curvatures can have at most two distinct multiplicities $m_1, m_2$, and Stolz showed that the pair $(m_1, m_2)$ must either be $(2,2)$, $(4,5)$, or be equal to the multiplicities of an isoparametric hypersurface of FKM-type, constructed by Ferus, Karcher and Münzner from orthogonal representations of Clifford algebras. In this paper, we prove that if the multiplicities satisfy $m_2 \geq 2m_1 - 1$, then the isoparametric hypersurface $M$ must be of FKM-type. Together with known results of Takagi for the case $m_1 = 1$, and Ozeki and Takeuchi for $m_1 = 2$, this handles all possible pairs of multiplicities except for 4 cases, for which the classification problem remains open.

1. Introduction

A hypersurface $M$ in a real space-form $M^n(c)$ of constant sectional curvature $c$ is said to be isoparametric if it has constant principal curvatures. An isoparametric hypersurface $M$ in $\mathbb{R}^n$ can have at most two distinct principal curvatures, and $M$ must be an open subset of a hyperplane, hypersphere or a spherical cylinder $S^k \times \mathbb{R}^{n-k-1}$. This was shown by Levi-Civita [18] for $n = 3$ and by B. Segre [27] for arbitrary $n$. Similarly, E. Cartan [3] proved that an isoparametric hypersurface $M$ in hyperbolic space $H^n$ can have at most two distinct principal curvatures, and $M$ must be either totally umbilic or else an open subset of a standard product $S^k \times H^{n-k-1}$ in $H^n$ (see also [8, pp.237-238]). However, Cartan [3]-[6] showed in a series of four papers written in the late 1930’s that the situation is much more interesting for isoparametric hypersurfaces in $S^n$. Cartan proved several general results and found

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examples with three and four distinct principal curvatures, as well as those with one or two. Despite the beauty of Cartan’s theory, it was relatively unnoticed for thirty years, until it was revived in the 1970’s by Nomizu [23]-[24] and Münzner [22].

Cartan showed that isoparametric hypersurfaces come as a family of parallel hypersurfaces, i.e., if \( x : M \rightarrow S^n \) is an isoparametric hypersurface, then so is any parallel hypersurface \( x_t \) at oriented distance \( t \) from the original hypersurface \( x \). However, if \( \lambda = \cot t \) is a principal curvature of \( M \), then \( x_t \) is not an immersion, since it is constant on the leaves of the principal foliation \( T_\lambda \), and \( x_t \) factors through an immersion of the space of leaves \( M/T_\lambda \) into \( S^n \). In that case, \( x_t \) is a focal submanifold of codimension \( m + 1 \) in \( S^n \), where \( m \) is the multiplicity of \( \lambda \). Münzner [22] showed that a parallel family of isoparametric hypersurfaces in \( S^n \) always consists of the level sets in \( S^n \) of a homogeneous polynomial \( F \) defined on \( \mathbb{R}^{n+1} \) satisfying certain differential equations which are listed at the beginning of Section 2. He showed that the level sets of \( F \) on \( S^n \) are connected, and thus any connected isoparametric hypersurface can be extended to a unique compact, connected isoparametric hypersurface. Münzner also showed that regardless of the number of distinct principal curvatures of \( M \), there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, and each isoparametric hypersurface in the family separates the sphere into two ball bundles over the two focal submanifolds. From this topological information, Münzner was able to prove his fundamental result that the number \( g \) of distinct principal curvatures of an isoparametric hypersurface in \( S^n \) must be 1, 2, 3, 4, or 6. As one would expect, classification results on isoparametric hypersurfaces have been dependent on the number of distinct principal curvatures.

Cartan classified isoparametric hypersurfaces with \( g \leq 3 \) principal curvatures. If \( g = 1 \), then \( M \) is umbilic and it must be a great or small sphere. If \( g = 2 \), then \( M \) must be a standard product of two spheres

\[
S^k(r) \times S^{n-k-1}(s) \subset S^n, \quad r^2 + s^2 = 1.
\]

In the case \( g = 3 \), Cartan [4] showed that all the principal curvatures must have the same multiplicity \( m = 1, 2, 4 \) or 8, and the isoparametric hypersurface must be a tube of constant radius over a standard Veronese embedding of a projective plane \( \mathbb{F}P^2 \) into \( S^{3m+1} \), where \( \mathbb{F} \) is the division algebra \( \mathbb{R}, \mathbb{C}, \mathbb{H} \) (quaternions), \( \mathbb{O} \) (Cayley numbers) for \( m = 1, 2, 4, 8 \), respectively. Thus, up to congruence, there is only one such family for each value of \( m \).

The classification of isoparametric hypersurfaces with four or six principal curvatures has stood as one of the outstanding problems in
submanifold geometry for some time, and it was listed as Problem 34 on Yau’s list of important open problems in geometry in 1992 (see [36] or [15]). In this paper, we will provide a partial solution to this classification problem in the case \( g = 4 \), but first we will describe the known results in the two cases.

In the case \( g = 6 \), there exists one homogeneous family with six principal curvatures of multiplicity one in \( S^7 \), and one homogeneous family with six principal curvatures of multiplicity two in \( S^{13} \) (see Miyaoka [20] for a description). These are the only known examples. Münzner showed that for \( g = 6 \), all of the principal curvatures must have the same multiplicity \( m \), and then Abresch [1] showed that \( m \) must be 1 or 2. In the case \( m = 1 \), Dorfmeister and Neher [10] showed in 1985 that an isoparametric hypersurface must be homogeneous, but it remains an open question whether this is true in the case \( m = 2 \).

For \( g = 4 \), there is a much larger and more diverse collection of known examples. Cartan produced examples of isoparametric hypersurfaces with four principal curvatures in \( S^5 \) and \( S^9 \). These examples are homogeneous, and have the property that all of the principal curvatures have the same multiplicity. Cartan asked if all isoparametric hypersurfaces must be homogeneous, and if there exists an isoparametric hypersurface whose principal curvatures do not all have the same multiplicity. Nomizu [23] generalized Cartan’s example in \( S^5 \) to produce a collection of isoparametric hypersurfaces whose principal curvatures have two distinct multiplicities \((1, k)\), for any positive integer \( k \), thereby answering Cartan’s second question in the affirmative. At approximately the same time as Nomizu’s work, Takagi and Takahashi [31] used the work of Hsiang and Lawson [17] on submanifolds of cohomogeneity two to determine all homogeneous isoparametric hypersurfaces of the sphere. Takagi and Takahashi showed that every homogeneous isoparametric hypersurface is a principal orbit of the isotropy representation of a rank two symmetric space, and they presented a complete list of examples. This list included some examples with 6 principal curvatures, as well as those with 1, 2, 3 or 4 distinct principal curvatures. In a separate paper, Takagi [30] proved that in the case \( g = 4 \), if one of the principal curvatures of \( M \) has multiplicity one, then \( M \) must be homogeneous.

In a two-part paper, Ozeki and Takeuchi [25] produced two infinite series of inhomogeneous isoparametric hypersurfaces with multiplicities \((3, 4k)\) and \((7, 8k)\), for any positive integer \( k \). They also classified isoparametric hypersurfaces for which one principal curvature has multiplicity two, proving that they must be homogeneous. In the process, Ozeki and Takeuchi developed a formulation of the Cartan-Münzner
polynomial $F$ in terms of the second fundamental forms of the focal submanifolds that is very useful in our work.

Next Ferus, Karcher and Münzner [13] used representations of Clifford algebras to construct for any positive integer $m_1$ an infinite series of isoparametric hypersurfaces with four principal curvatures having multiplicities $(m_1, m_2)$, where $m_2$ is nondecreasing and unbounded in each series. In fact, $m_2 = k\delta(m_1) - m_1 - 1$, where $\delta(m_1)$ is the positive integer such that the Clifford algebra $C_{m_1-1}$ has an irreducible representation on $\mathbb{R}^{\delta(m_1)}$ (see [2]), and $k$ is any positive integer for which $m_2$ is positive. Isoparametric hypersurfaces obtained by this construction of Ferus, Karcher and Münzner are said to be of FKM-type. The FKM-series with multiplicities $(3, 4k)$ and $(7, 8k)$ are precisely those constructed by Ozeki and Takeuchi. For isoparametric hypersurfaces of FKM-type, one of the focal submanifolds is always a Clifford-Stiefel manifold (see Pinkall-Thorbergsson [26]).

The set of FKM-type isoparametric hypersurfaces contains all known examples with $g = 4$ with the exception of two homogeneous examples, with multiplicities $(m_1, m_2)$ equal to $(2, 2)$ and $(4, 5)$ (see [25, part II, p.27] for more detail on these two exceptions). Over the years, many restrictions on the multiplicities were found by Münzner [22], Abresch [1], Grove and Halperin [16], Tang [32] and Fang [12]. This series of papers culminated in the recent work of Stolz [29], who showed that the multiplicities of an isoparametric hypersurface with $g = 4$ must be the same as those in the known examples of Ferus, Karcher and Münzner or the two homogeneous exceptions. This certainly adds weight to the conjecture that the known examples are actually the only isoparametric hypersurfaces with $g = 4$. In this paper, we prove that this conjecture is true, if the two multiplicities satisfy $m_2 \geq 2m_1 - 1$. Specifically, we prove (see Theorem 47):
isoparametric hypersurface to be of FKM-type. The second part shows that these conditions are satisfied if \( m_2 \geq 2m_1 - 1 \).

Next we will provide a detailed outline of the paper. For more information on isoparametric hypersurfaces and the extensive theory of isoparametric submanifolds of codimension greater than one in the sphere, which was introduced by Carter and West [7] and Terng [33], the reader is referred to the excellent survey article by Thorbergsson [35], who proved that all isoparametric submanifolds of codimension greater than one in the sphere are homogeneous [34].

We think of an isoparametric hypersurface as an immersion \( \bar{x} : M^{n-1} \to S^n \). About any point of \( M \) there is a neighborhood \( U \) on which there is defined an orthonormal frame field \( \bar{x}, \bar{e}_0, e_a, e_p, e_a, e_\mu \) for which \( \bar{e}_0 \) is normal to the hypersurface and the other sets of vectors are principal directions for the four respective principal curvatures of \( \bar{x} \). The index range of \( a, p \) has length \( m \), and that of \( \alpha, \mu \) has length \( N \), where \( m = m_1 \) and \( N = m_2 \) are the multiplicities for our isoparametric hypersurface. The dual coframe on \( U \) is the set of 1-forms \( \theta^a, \theta^p, \theta^\alpha, \theta^\mu \) defined on \( U \) by the equation (sum on repeated indices)

\[
d\bar{x} = \theta^a e_a + \theta^p e_p + \theta^\alpha e_\alpha + \theta^\mu e_\mu
\]

The curvature surfaces are the integral submanifolds of the distribution obtained by setting any three sets of these forms equal to zero. The Levi-Civita connection forms of a curvature surface are given, essentially, by the forms \( \theta^a = de_a \cdot e_b, \theta^p = de_a \cdot e_p, \) etc. The second fundamental tensors of the focal submanifolds are given in terms of our frame field by the four sets of tensors \( F^\mu_{aa}, F^\mu_{ap}, F^\mu_{pa}, \) and \( F^\alpha_{pa} \) defined in (4.18) in which the coframe field \( \omega^a, \omega^p, \omega^\alpha, \omega^\mu \) is defined in (4.13) as constant multiples of \( \theta^a, \theta^p, \theta^\alpha, \theta^\mu \), respectively. We derive the identities imposed on these tensors and their derivatives by the Maurer-Cartan structure equations of the orthogonal group \( O(n+1) \), the isometry group of \( S^n \).

If our isoparametric hypersurface is of FKM-type, then a simple calculation shows that the following equations hold for an appropriate choice of the Darboux frame field.

\[
F^\mu_{a a+m} = F^\mu_{a a}
\]

\[
F^\alpha_{b+m a} + F^\alpha_{a+m b} = 0
\]

\[
F^\mu_{b+m a} + F^\mu_{a+m b} = 0
\]

\[
\theta^a_b - \theta^a_{b+m} = L^a_{bc}(\omega^c + \omega^{c+m}), \quad L^a_{bc} = -L^b_{ac} = -L^a_{cb}
\]

where \( a, b, c = 1, \ldots, m \) and \( a + m, b + m \) run through the range of the indices \( p, q \). The matrices of the operators of the Clifford system in terms of our frame field have as entries certain constants and the
functions $F^\mu_{\alpha a}$, $F^\mu_{\alpha p}$, $F^\mu_{p\alpha}$, $F^\alpha_{p\alpha}$, and $L^a_{bc}$. Thus, using these matrices, we can define these operators for an arbitrary isoparametric hypersurface. If equations (1.1)-(1.4) hold for the isoparametric hypersurface, then by an elementary, but extremely long, calculation we show that these operators form a Clifford system whose FKM construction produces the given isoparametric hypersurface. This calculation is contained in the proof of Theorem 24.

In Proposition 19 we prove that (1.1) implies (1.2)-(1.4) on $U$ provided that $\tilde{x}$ satisfies the spanning property (Definition 8), which is:

(a). There exists a vector $\sum_a x_a e_a$ such that

$$\left\{ \sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_a y_\mu e_a : (y_\mu) \in \mathbb{R}^N \right\} = \text{span} \{ e_1, \ldots, e_m \}$$

(b). There exists a vector $\sum_\mu y_\mu e_\mu$ such that

$$\left\{ \sum_{a,\alpha,\mu} F^\mu_{\alpha a} x_a y_\mu e_\mu : (x_\alpha) \in \mathbb{R}^N \right\} = \text{span} \{ e_1, \ldots, e_m \}$$

Combining these results, we see that if an isoparametric hypersurface satisfies the spanning property and (1.1) on $U$, then it is of FKM-type. The next step is to see when (1.1) will be true.

The parallel hypersurface at an oriented distance $t$ from $\tilde{x}$ is given by $x = \cos t \tilde{x} + \sin t \tilde{e}_0$. Its unit normal vector is $e_0 = -\sin t \tilde{x} + \cos t \tilde{e}_0$ and its principal directions are still given by the remaining vectors in the frame field. At some value of $t$ the rank of $x$ is less than $n - 1$, in which case the image of $x$ is a focal submanifold of the isoparametric family. Any multiple of $\pi/4$ added to this value of $t$ again gives a focal submanifold. From Münzner's result that there are only two focal submanifolds, it follows that as $t$ changes by a multiple of $\pi/2$, we return to the same focal submanifold. If $x$ is a focal submanifold, then we may assume that $e_0, e_a$ is a normal frame field along $x$ and the vectors $e_p$, $e_\alpha$, $e_\mu$ are the principal vectors for the second fundamental form $II_{e_a}$, of principal curvatures 0, 1 and $-1$, respectively. Moving a distance $t = \pi/2$ from $x$ along the geodesic in the direction of $e_0$, we arrive at $e_0$, which must then be a position vector on the same focal submanifold. At $e_0$, the normal frame field is $x, e_p$, and the principal vectors, of principal curvatures 0, 1 and $-1$ are $e_a, e_\alpha$ and $e_\mu$, respectively.

There is a simple relationship between the four sets of tensors at $e_0$, denoted with the same letters barred, and these tensors at $x$. For our purposes, the most important is

$$\tilde{F}^\mu_{\alpha a} = F^\mu_{\alpha a + m}$$
Use these tensors to define real bihomogeneous polynomials

\[ p_a(x, y) = \sum_{\alpha, \mu} F^\mu_{\alpha a} x_\alpha y_\mu, \quad \bar{p}_a(x, y) = \sum_{\alpha, \mu} \bar{F}^\mu_{\alpha a} x_\alpha y_\mu \]

In Proposition 11 we prove that if \( x \) satisfies the spanning property on \( U \) and if at each point of \( U \) the \( \bar{p}_a \) are contained in the ideal \( I \) generated by \( p_1, \ldots, p_m \) in the polynomial ring \( \mathbb{R}[x_\alpha, y_\mu] \), then the frame field can be chosen so that (1.1) holds on \( U \).

The key to linking the set of polynomials \( \bar{p}_a \) with the set of polynomials \( p_a \) comes from a formula for the isoparametric function derived by Ozeki and Takeuchi [25] (recorded in (10.1) below). In Proposition 27 (see also Proposition 28) we use this formula to prove that the zero locus of \( p_1, \ldots, p_m \) in \( \mathbb{R}P^{N-1} \times \mathbb{R}P^{N-1} \) is identical with that of \( \bar{p}_1, \ldots, \bar{p}_m \).

Algebraic geometers have developed a substantial body of information about the relationship between two polynomial ideals whose zero varieties coincide. Let \( I \) be the ideal generated by \( p_1, \ldots, p_m \) in the polynomial ring \( \mathbb{R}[x_\alpha, y_\mu] \) and let \( I^C \) be the ideal they generate in the polynomial ring \( \mathbb{C}[x_\alpha, y_\mu] \). For \( 1 \leq s \leq m \), define the affine bi-cones

\[ V_s = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : p_a(x, y) = 0, \ a = 1, \ldots, s\} \]
\[ V_s^C = \{(x, y) \in \mathbb{C}^N \times \mathbb{C}^N : p_a(x, y) = 0, \ a = 1, \ldots, s\} \]

We denote \( V_m \) and \( V_m^C \), which are in fact what we are after, by \( V_I \) and \( V_I^C \), respectively. Let \( J_s \) be the complex subvariety of \( V_s^C \) where the Jacobian matrix of \( p_1, \ldots, p_s \) is of rank less than \( s \). In our Classification Theorem 47 we prove the following. Fix a point in \( U \). Assume \( N \geq m + 2 \). If the codimension of \( J_s \) is greater than 1 in \( V_s^C \) for all \( s \leq m \), then, at the point, through an inductive procedure, we establish

(I) \( p_1, \ldots, p_m \) form a regular sequence in \( \mathbb{C}[x_\alpha, y_\mu] \)
(II) \( \dim \mathbb{R} V_I = \dim \mathbb{C} V_I^C \)
(III) \( I^C \) is a prime ideal of codimension \( m \)
(IV) The spanning property holds for \( x \).

The primeness (more generally, reducedness) of \( I^C \) is precisely the condition which allows us to conclude that the \( \bar{p}_a \in I \).

The final step in our argument is then provided by Proposition 46 which states that for \( N \geq m + 2 \), if \( N \geq 2m \), then indeed \( \text{codim} (J_s) \geq 2 \) for all \( s \leq m \) at every point of \( U \), so that \( I^C \) is prime; as a result, if \( N = 2m - 1 \), then \( I^C \) is a reduced ideal. The proof of this estimate requires a detailed analysis of the second fundamental forms \( II_{e_a} \) of \( x \).

In the case \( m = 1 \), we give a simpler proof that \( M \) is of FKM-type,
thereby providing another proof of Takagi’s result. Our approach also recovers Ozeki-Takeuchi’s result when $m = 2$ and $N \geq 3$.

The paper is very much self-contained, and we have made an effort to make the exposition as clear as possible. We would like to thank N. Mohan Kumar for substantial help with the algebraic geometry and John Little for his comments on previous versions of this paper. We are grateful to the referee, whose many helpful comments have improved the exposition and quality of the paper.

2. SECOND ORDER FRAMES

An immersed connected oriented hypersurface $\mathbf{x} : M^{n-1} \to S^n$ is called isoparametric if $\mathbf{x}$ has constant principal curvatures. Such a hypersurface always occurs as part of a family, the level surfaces of an isoparametric function $f$, which is a smooth function on $S^n$ such that $|\nabla f|^2 = a(f)$ and $\Delta f = b(f)$, for some smooth functions $a, b : \mathbb{R} \to \mathbb{R}$.

Denote the principal curvatures of $\mathbf{x}$ by $k_i$, with multiplicity $m_i$, for $i = 1, \ldots, g$, and assume that $k_1 > \cdots > k_g$. Münzner [22, part I] showed that the multiplicities satisfy $m_i = m_{i+2}$ (subscripts mod $g$). He then showed that the isoparametric function $f$ must be the restriction to $S^n$ of a homogeneous polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ of degree $g$ satisfying the differential equations

$$
|\text{grad } F|^2 = g^2 r^{2g-2}, \quad r = |\mathbf{x}|
$$

$$
\Delta F = \frac{m_2 - m_1}{2} g^2 r^{g-2}
$$

where $m_1$ and $m_2$ are the two (possibly equal) multiplicities. The polynomial $F$ is called the Cartan-Münzner polynomial of the family of isoparametric hypersurfaces, and $F$ takes values between $-1$ and $1$ on the sphere $S^n$. For $-1 < t < 1$, the level set $F^{-1}(t)$ in $S^n$ is one of the isoparametric hypersurfaces in the family. The level sets $M_+ = F^{-1}(1)$ and $M_- = F^{-1}(-1)$ are the two focal submanifolds of the family, having codimensions $m_1 + 1$ and $m_2 + 1$ in $S^n$, respectively.

We now develop the local geometry of isoparametric hypersurfaces using the method of moving frames in the sphere. In the process, we will reprove some of the results obtained by Münzner, although this is not our primary goal.

We assume now that $g = 4$, even though many of the results in Sections 2 – 4 have analogues for arbitrary values of $g$. Let $\mathbf{e}_0$ be the unit normal vector field along $\mathbf{x}$ defining the orientation of $M$. Any point of $M$ has an open neighborhood $U$ on which there exists a Darboux frame field $\mathbf{x}, e_1, \mathbf{e}_0 : U \to SO(n+1)$, $1 \leq i \leq n - 1$, for which
each vector $e_i$ is a principal direction. We adopt the index ranges

\begin{align}
  i, j, k & \in \{1, \ldots, n-1\} \\
  a, b, c & \in \{1, \ldots, m_1\}, \quad p, q, r \in \{m_1 + 1, \ldots, m_1 + m_3\} \\
  \alpha, \beta, \gamma & \in \{m_1 + m_3 + 1, \ldots, m_1 + m_2 + m_3\} \\
  \mu, \nu, \sigma & \in \{m_1 + m_2 + m_3 + 1, \ldots, n-1\}
\end{align}

(2.1)

Arrange the frame so that the $e_\alpha$ span the principal space for $k_1$, the $e_\beta$ span the principal space for $k_2$, the $e_\gamma$ span the principal space for $k_3$, and the $e_\mu$ span the principal space for $k_4$. We shall call such a Darboux frame field

\begin{equation}
  \mathbf{x}, e_\alpha, e_\beta, e_\gamma, e_\mu, \tilde{e}_0
\end{equation}

(2.2)

on $U$ a second order frame field along $\mathbf{x}$, (a first order Darboux frame field is one for which $\tilde{e}_0$ is normal and the remaining vectors are tangent, but not necessarily principal directions). For such a frame field

\begin{equation}
  d\mathbf{x} = \theta^i e_i \quad \text{and} \quad de_i = \theta^i_j e_j - \theta^i \mathbf{x} + \theta^0_i \tilde{e}_0
\end{equation}

(2.3)

where $\theta^i$, $\theta^0_i = -\theta^i_0$, $\theta^i_j = -\theta^j_i$ are 1-forms on $U$ and $\theta^1, \ldots, \theta^{n-1}$ is an orthonormal coframe field on $U$ with respect to the metric induced by $\mathbf{x}$ on $M$. Notice that $\theta^0 = d\mathbf{x} \cdot \tilde{e}_0 = 0$. We use the Einstein summation convention unless the contrary is stated explicitly. This means that repeated indices in a product are to be summed over the range defined in (2.1). In some instances the repeated indices are both up, or both down, but still they are to be summed as in the standard case of one up and one down. The 1-forms in (2.3) satisfy the Maurer-Cartan structure equations of $SO(n+1)$

\begin{align}
  d\theta^i & = -\theta^i_j \wedge \theta^j \\
  d\theta^0_i & = -\theta^0_j \wedge \theta^i_j \\
  d\theta^j & = \theta^i \wedge \theta^i_j - \theta^i_0 \wedge \theta^0_j - \theta^j_k \wedge \theta^k_j
\end{align}

(2.4)

We also have

\begin{equation}
  d\tilde{e}_0 = \theta^i_0 e_i
\end{equation}

(2.5)

where the 1-forms $\theta^i_0 = -\theta^0_i$ are linear combinations of the coframe forms, namely

\begin{equation}
  \theta^0_i = h_{ij} \theta^j
\end{equation}

(2.6)

where these coefficient functions on $U$ satisfy $h_{ij} = h_{ji}$ as a consequence of taking the exterior derivative of the equation $\theta^0 = 0$. The second fundamental form of $\mathbf{x}$ is

\begin{equation}
  \widetilde{II} = -d\mathbf{x} \cdot d\tilde{e}_0 = h_{ij} \theta^i \theta^j
\end{equation}

(2.7)
Having chosen the $e_i$ to be principal vectors, we know that the symmetric matrix $h_{ij}$ is a diagonal matrix. In fact, we have

\begin{equation}
(2.8) \quad \theta^0_a = k_1 \theta^a, \quad \theta^0_p = k_3 \theta^p, \quad \theta^0_\alpha = k_2 \theta^\alpha, \quad \theta^0_\mu = k_4 \theta^\mu
\end{equation}

Set $\theta^i_j = \sum h^i_{jk} \theta^k$, where the smooth function coefficients satisfy $h^i_{jk} = -h^i_{kj}$, for all $i, j, k = 1, \ldots, n - 1$. Take the exterior differential of equations (2.8), using the structure equations of $SO(n + 1)$, to find

\begin{equation}
(2.9) \quad \theta^p_a = h^p_{a\alpha} \theta^\alpha + h^p_{a\mu} \theta^\mu, \quad \text{since} \quad h^p_{ab} = 0 = h^p_{aq}
\end{equation}

\begin{equation}
\theta^a_p = h^a_{p\alpha} \theta^\alpha + h^a_{p\mu} \theta^\mu, \quad \text{since} \quad h^a_{ab} = -h^a_{ab} = 0 = h^a_{a\beta}
\end{equation}

\begin{equation}
\theta^\alpha_a = h^\alpha_{a\mu} \theta^\mu + h^\alpha_{a\alpha} \theta^\alpha, \quad \text{since} \quad h^\alpha_{ab} = -h^\alpha_{ab} = 0 = h^\alpha_{a\nu}
\end{equation}

\begin{equation}
\theta^\mu_p = h^\mu_{p\alpha} \theta^\alpha + h^\mu_{p\mu} \theta^\mu, \quad \text{since} \quad h^\mu_{pq} = -h^\mu_{pq} = 0 = h^\mu_{p\nu}
\end{equation}

\begin{equation}
\theta^\alpha_p = h^\alpha_{p\alpha} \theta^\alpha + h^\alpha_{p\mu} \theta^\mu, \quad \text{since} \quad h^\alpha_{pa} = -h^\alpha_{pa} = 0 = h^\alpha_{a\nu}
\end{equation}

and the coefficient functions further satisfy

\begin{equation}
(2.10) \quad (k_3 - k_1)h^p_{a\alpha} = (k_2 - k_1)h^\alpha_{ap} = (k_2 - k_3)h^\alpha_{pa}
\end{equation}

\begin{equation}
(k_3 - k_1)h^a_{p\mu} = (k_4 - k_1)h^\mu_{ap} = (k_4 - k_3)h^\mu_{pa}
\end{equation}

\begin{equation}
(k_2 - k_1)h^\alpha_{a\mu} = (k_4 - k_1)h^\mu_{a\alpha} = (k_4 - k_2)h^\mu_{aa}
\end{equation}

\begin{equation}
(k_2 - k_3)h^\alpha_{p\mu} = (k_4 - k_3)h^\mu_{p\alpha} = (k_4 - k_2)h^\mu_{ap}
\end{equation}

At a point of $M$ the set of principal vectors for a principal curvature $k_i$ is a subspace of dimension $m_i$, defined by the equations $\theta^i_j = 0$, for all $j$ not in the range of the given principal curvature. This $m_i$-plane distribution on $M$ is called a curvature distribution on $M$.

**Lemma 1.** The curvature distributions are completely integrable. Their integral submanifolds are called curvature surfaces. A curvature surface corresponding to $k_j$ is totally geodesic in $M$ and its induced metric has constant sectional curvature $1 + k_j^2$.

**Proof.** This is a simple application of the structure equations and the first three equations in (2.9).

**Remark 2.** One can show that each curvature surface corresponding to $k_j$ is also totally geodesic in the curvature sphere of $M$ corresponding to $k_j$ (see Theorems 4.11 - 4.13 of [8, pp. 149 - 150]).

Additional conditions are imposed by the structure equations on the coefficients upon the exterior differentiation of equations (2.9).
3. Parallel hypersurfaces

Let $\tilde{x}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0$ be a second order frame field (2.2) along $\tilde{x}$ on $U$. We may arrange to have $k_1 > k_2 > k_3 > k_4$. It will be convenient to set $k_i = \cot s_i$, for $i = 1, \ldots, 4$, where $0 < s_1 < s_2 < s_3 < s_4 < \pi$.

For any fixed real number $t$, let

\begin{equation}
\tilde{x} = \cos t \tilde{x} + \sin t \tilde{e}_0
\end{equation}

From (2.3), (2.5) and (2.8) we have

\begin{equation}
d \tilde{x} = (\cos t - \sin t \cot s_1) \theta^a e_a + (\cos t - \sin t \cot s_3) \theta^p e_p
+ (\cos t - \sin t \cot s_2) \theta^a e_\alpha + (\cos t - \sin t \cot s_4) \theta^\mu e_\mu
\end{equation}

We conclude that $\tilde{x}$ is an immersion of $M$ except when $t \equiv s_i \mod \pi$, for some $i = 1, 2, 3, 4$. Suppose $t$ is not one of these exceptional values. Then the unit normal vector field along $\tilde{x}$ preserving the orientation of $M$ is

\begin{equation}
e_0 = -\sin t \tilde{x} + \cos t \tilde{e}_0
\end{equation}

and again from (2.3), (2.5) and (2.8) we have

\begin{equation}
d e_0 = -(\sin t + \cos t \cot s_1) \theta^a e_\alpha - (\sin t + \cos t \cot s_3) \theta^p e_\mu
- (\sin t + \cos t \cot s_2) \theta^a e_\alpha - (\sin t + \cos t \cot s_4) \theta^\mu e_\mu
\end{equation}

Since $(\sin t + \cos t \cot s)/(\cos t - \sin t \cot s) = \cot(s - t)$, for any $s$ and $t$, we find that the second fundamental form of $\tilde{x}$ is

\begin{equation}
II = -d \tilde{x} \cdot d e_0
= \cot(s_1 - t) \omega^a \omega^a + \cot(s_3 - t) \omega^p \omega^p
+ \cot(s_2 - t) \omega^a \omega^\alpha + \cot(s_4 - t) \omega^\mu \omega^\mu
\end{equation}

We conclude that the principal curvatures of $\tilde{x}$ are constant, equal to $\cot(s_i - t)$ with multiplicity $m_i$, for $i = 1, 2, 3, 4$, and that

\begin{equation}
\tilde{x}, e_a, e_p, e_\alpha, e_\mu, e_0
\end{equation}

is a second order frame field along $\tilde{x}$ on $U$.

4. Focal submanifolds

We consider now what happens when $t$ is one of the exceptional values. To be specific, suppose that $t = s_1$. Then $\tilde{x}$ is defined in (3.1) and $e_0$ is defined in (3.3) with $t = s_1$. For the frame field (3.6) along $\tilde{x}$ on $U$, equation (3.2) becomes

\begin{equation}
d \tilde{x} = \omega^p e_p + \omega^a e_\alpha + \omega^\mu e_\mu
\end{equation}
whose rank is \( n - 1 - m_1 \) at every point of \( M \) and where
\[
\omega^p = \frac{\sin(s_3 - s_1)}{\sin s_3} \theta^p, \quad \omega^a = \frac{\sin(s_2 - s_1)}{\sin s_2} \theta^a, \quad \omega^\mu = \frac{\sin(s_4 - s_1)}{\sin s_4} \theta^\mu
\]

Therefore, the image \( x(M) \) is a submanifold of codimension \( m_1 + 1 \) in \( S^n \). It is called the focal submanifold for the principal curvature \( \cot s_1 \). In the same way, there are focal submanifolds for each of the principal curvatures. For a point \( v \in x(M) \), the set \( L = x^{-1}\{v\} \) is a curvature surface of \( x \) for the principal curvature \( \cot s_1 \). Restricted to this curvature surface, the forms \( \theta^\alpha \) give a coframe field on it.

If \( e_0 \) is defined by \((3.3)\), then \((4.1)\) shows that \( x, e_p, e_a, e_\mu, e_a, e_0 \) is a Darboux frame field along \( x \), with \( e_p, e_a, e_\mu \) tangent and \( e_0, e_a \) normal vectors. Take a point \( p \) in the curvature surface \( L \) and let \( N \) denote the normal space to \( x \) at \( p \). Let \( S^{m_1} \) denote the unit sphere in \( N \). The next lemma shows that \( e_0(L) \) covers an open neighborhood of \( e_0(p) \) in this sphere.

**Lemma 3.** The rank of \( e_0 : L \to S^{m_1} \) is \( m_1 \) at every point of the curvature surface \( L \). Therefore, \( e_0(L) \) covers an open neighborhood of \( e_0(p) \) in \( S^{m_1} \).

**Proof.** Consider the frame field \( e_0, e_a, x, e_p, e_a, e_\mu \) along \( e_0 \) on \( L \). Since \( \theta^p, \theta^a \) and \( \theta^\mu \) are all zero pulled back to \( L \), it follows from \((2.9)\) that \( \theta^p_0, \theta^a_0 \) and \( \theta^\mu_0 \) are also zero pulled back to \( L \). Therefore, restricted to \( L \), and using \((2.8)\), in which now \( k_1 = \cot s_1 \), we have
\[
d e_0 = -\cos s_1 \theta^a e_a + \cos s_1 \theta^\mu e_\mu = -\csc s_1 \theta^a e_a
\]
which has rank equal to \( m_1 \) at every point of \( L \). \qed

We can now calculate the second fundamental form of the submanifold \( x \) at the point \( x(p) = v \) with respect to any unit normal vector there.

**Lemma 4.** At any point of \( M \) and with respect to any unit normal vector at the point, the principal curvatures of the focal submanifold \( x \) are
\[
\cot(s_2 - s_1), \quad \cot(s_3 - s_1), \quad \cot(s_4 - s_1)
\]
with multiplicities \( m_2, m_3, m_4 \), respectively.

**Proof.** From \((3.4)\) we have for \( t = s_1 \)
\[
d e_0 = -\frac{1}{\sin s_1} \theta^a e_a - \frac{\cos(s_3 - s_1)}{\sin s_3} \theta^\mu e_p
\]
\[
- \frac{\cos(s_2 - s_1)}{\sin s_2} \theta^\alpha e_a - \frac{\cos(s_4 - s_1)}{\sin s_4} \theta^\mu e_\mu
\]
Combining this with (4.2) we have for the second fundamental form at $p$ with respect to the normal vector $e_0$

$$II_{e_0} = -d\mathbf{x} \cdot de_0 = \cot(s_3 - s_1)\omega^p\omega^p + \cot(s_2 - s_1)\omega^\alpha\omega^\alpha + \cot(s_4 - s_1)\omega^\mu\omega^\mu$$

where $\omega^p, \omega^\alpha, \omega^\mu$, defined in (4.3), form an orthonormal coframe with respect to the metric induced by $\mathbf{x}$ on the focal submanifold for the principal curvature $\cot s_1$. By Lemma 3 we know that $e_0(L)$ covers some open subset of the unit sphere in the normal space to $\mathbf{x}$ at $p$. Since the characteristic polynomial of $II_n$ is an analytic function of $n$ in the unit sphere of the normal space, it follows that the eigenvalues of $II_n$ must be given by (4.4) for every unit normal vector at $p$. (See [8, Proof of Corollary 2.2 on p. 249]).

Münzner [22, Part I] proved Lemma 4 and used it to prove the following important consequence (see also [8, p. 249]).

**Corollary 5.** The angles $s_i = s_1 + (i - 1)\pi/4$, for $i = 2, 3, 4$ and the multiplicities satisfy $m_1 = m_3$ and $m_2 = m_4$. To simplify the notation we set $m_1 = m_3 = m$ and $m_2 = m_4 = N$.

Given these facts, our index conventions (2.1) become

$$i, j, k \in \{1, \ldots, n - 1\}, \quad a, b, c \in \{1, \ldots, m\}$$

\begin{align*}
(4.6) \quad & p, q, r \in \{m + 1, \ldots, 2m\}, \quad \alpha, \beta, \gamma \in \{2m + 1, \ldots, 2m + N\} \\
& \mu, \nu, \sigma \in \{2m + N + 1, \ldots, n - 1\}
\end{align*}

so that $2m + 2N = n - 1$, and $n$ must be odd. Combining Lemma 4 and Corollary 5 yields the following.

**Corollary 6.** At any point of $M$ and with respect to any unit normal vector of $\mathbf{x}$ at the point, the principal curvatures of $\mathbf{x}$ are

$$k_1, \quad 0, \quad -1$$

with multiplicities $N, m$ and $N$, respectively.

In the light of Corollary 5, the principal curvatures $k_i = \cot s_i$ of $\mathbf{x}$ satisfy

$$k_2 = \frac{k_1 - 1}{k_1 + 1}, \quad k_3 = -\frac{1}{k_1}, \quad k_4 = \frac{1 + k_1}{1 - k_1}$$
We will have occasion to use the following differences of these principal curvatures.

\begin{equation}
\begin{aligned}
k_2 - k_1 &= -\frac{1 + k_1^2}{1 + k_1}, \quad k_3 - k_1 = -\frac{1 + k_1^2}{k_1} \\
k_4 - k_1 &= \frac{1 + k_1^2}{1 - k_1}, \quad k_3 - k_2 = -\frac{1 + k_1^2}{k_1(1 + k_1)} \\
k_4 - k_2 &= 2\frac{1 + k_1^2}{1 - k_1^2}, \quad k_4 - k_3 = \frac{1 + k_1^2}{k_1(1 - k_1)}
\end{aligned}
\end{equation}

We use equations (4.9) to rewrite equations (2.10) as

\begin{equation}
\begin{aligned}
h_{a\alpha} &= -\frac{1}{1 + k_1} h_{pa}^{\alpha}, \quad h_{\alpha p}^{\alpha} = -\frac{1}{k_1} h_{pa}^{\alpha} \\
h_{a\mu}^{\mu} &= \frac{1}{k_1 - 1} h_{pa}^{\mu}, \quad h_{\alpha p}^{\mu} = \frac{1}{k_1} h_{pa}^{\mu} \\
h_{a\alpha}^{\alpha} &= \frac{2}{k_1 - 1} h_{aa}^{\alpha}, \quad h_{\alpha a}^{\alpha} = \frac{2}{1 + k_1} h_{aa}^{\alpha} \\
h_{p\mu}^{\mu} &= \frac{2k_1}{1 - k_1} h_{ap}^{\mu}, \quad h_{\mu pa}^{\mu} = \frac{2k_1}{1 + k_1} h_{ap}^{\mu}
\end{aligned}
\end{equation}

Now, with \( s_i = s_1 + (i - 1)\pi / 4 \), equation (4.1) takes the form

\begin{equation}
dx = \frac{1}{\sin s_3} \theta^p e_p + \frac{1}{\sqrt{2} \sin s_2} \theta^\alpha e_\alpha + \frac{1}{\sqrt{2} \sin s_1} \theta^\mu e_\mu
\end{equation}

and with \( t = s_1 \) equation (3.4) becomes

\begin{equation}
\begin{aligned}
de_0 &= -\frac{1}{\sin s_1} \theta^a e_a - \frac{1}{\sqrt{2} \sin s_2} \theta^\alpha e_\alpha + \frac{1}{\sqrt{2} \sin s_4} \theta^\mu e_\mu
\end{aligned}
\end{equation}

If we define a new coframe field on \( U \subset M \) by

\begin{equation}
\begin{aligned}
\omega^\alpha &= -\frac{1}{\sin s_1} \theta^\alpha, \quad \omega^\alpha = \frac{1}{k_1 \sin s_1} \theta^\alpha \\
\omega^\alpha &= \frac{1}{(1 + k_1) \sin s_1} \theta^\alpha, \quad \omega^\alpha = \frac{1}{(k_1 - 1) \sin s_1} \theta^\alpha
\end{aligned}
\end{equation}

then, because

\begin{equation}
\begin{aligned}
\sin s_2 &= \frac{1 + k_1}{\sqrt{2}} \sin s_1, \quad \sin s_3 = k_1 \sin s_1, \quad \sin s_4 = \frac{k_1 - 1}{\sqrt{2}} \sin s_1
\end{aligned}
\end{equation}

equations (4.11) and (4.12) become

\begin{equation}
\begin{aligned}
dx &= \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu \\
\begin{aligned}
de_0 &= \omega^a e_a - \omega^\alpha e_\alpha + \omega^\mu e_\mu
\end{aligned}
\end{aligned}
\end{equation}

One conclusion we can draw from (4.15) is that

\begin{equation}
\begin{aligned}
x, e_0, e_\alpha, e_p, e_\alpha, e_\mu
\end{aligned}
\end{equation}
is a Darboux frame field along $\mathbf{x}$ on $U$, with $e_0, e_\alpha$ normal vectors and $e_p, e_\alpha, e_\mu$ tangent vectors spanning the principal spaces of curvature 0, 1 and $-1$, respectively of $II_{e_0}$. We shall call this a *second order frame field* along the focal submanifold $\mathbf{x}$ on $U$. For each point of $U$, define linear subspaces of $\mathbb{R}^{n+1}$ by

\begin{equation}
V_+ = \text{span}\{e_\alpha\}, \quad V_- = \text{span}\{e_\mu\}, \quad V_0 = \text{span}\{e_p\}
\end{equation}

These are the +1, −1 and 0 principal curvature spaces, respectively, for the normal vector $e_0$ at this point. If we express the Maurer-Cartan forms (2.9) in terms of our coframe field (4.13) as

\begin{equation}
\theta^p_a = \sum_a F^a_{pa} \omega^a - \sum_\mu F^\mu_{pa} \omega^\mu, \quad \theta^\alpha_a = \sum_\mu F^\alpha_{pa} \omega^\mu - 2 \sum_\mu F^\mu_{a\alpha} \omega^\mu
\end{equation}

\begin{equation}
\theta^\mu_p = \sum_a F^\mu_{pa} \omega^a - 2 \sum_\mu F^\mu_{ap} \omega^\mu, \quad \theta^a_\alpha = - \sum_\mu F^\mu_{pa} \omega^\mu - 2 \sum_\alpha F^\mu_{a\alpha} \omega^\alpha
\end{equation}

\begin{equation}
\theta^\mu_\alpha = \sum_a F^\mu_{pa} \omega^a + 2 \sum_\alpha F^\mu_{ap} \omega^\alpha, \quad \theta^\alpha_\mu = \sum_a F^\alpha_{a\alpha} \omega^a + \sum_\mu F^\mu_{a\mu} \omega^\mu
\end{equation}

then comparison with (2.9), using (4.10) and (4.13), gives

\begin{equation}
F^\alpha_{pa} = -h^\alpha_{pa} \sin s_1, \quad F^\mu_{pa} = -h^\mu_{pa} \sin s_1
\end{equation}

\begin{equation}
F^\alpha_{a\alpha} = -h^\mu_{a\alpha} \sin s_1, \quad F^\mu_{a\alpha} = h^\mu_{a\alpha} \cos s_1
\end{equation}

Notice that the distribution obtained by setting any three sets of $\{\omega^a\}$, $\{\omega^p\}$, $\{\omega^\alpha\}$ and $\{\omega^\mu\}$ equal to zero is completely integrable and its integral submanifolds are the respective curvature surfaces.

Equations (2.3) become, for the Darboux frame field (4.16),

\begin{equation}
d\mathbf{x} = \omega^p e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu
d e_0 = \omega^\alpha e_\alpha - \omega^\mu e_\mu
d e_\alpha = -\omega^\alpha e_0 + \theta^b_\alpha e_b + \theta^q_\alpha e_q + \theta^e_\alpha e_e + \theta^\mu_\alpha e_\mu
\end{equation}

\begin{equation}
d e_p = -\omega^\alpha \mathbf{x} + \theta^b_\alpha e_b + \theta^q_\alpha e_q + \theta^e_\alpha e_e + \theta^\mu_\alpha e_\mu
d e_\alpha = -\omega^\alpha \mathbf{x} + \omega^\alpha e_0 + \theta^b_\alpha e_b + \theta^q_\alpha e_q + \theta^e_\alpha e_e + \theta^\mu_\alpha e_\mu
d e_\mu = -\omega^\mu \mathbf{x} - \omega^\mu e_0 + \theta^b_\mu e_b + \theta^q_\mu e_q + \theta^e_\mu e_e + \theta^\alpha_\mu e_\alpha + \theta^\beta_\mu e_\beta + \theta^\gamma_\mu e_\gamma
\end{equation}

The Cartan-Münzner polynomial $F : \mathbb{R}^{n+1} \to \mathbb{R}$ defining the isoparametric function $f = F|_{S^n} : S^n \to [-1, 1]$ has ±1 as the only two singular values, and focal points at a distance $\pi/2$ along a normal geodesic from each other lie on the same focal submanifold. If our second order Darboux frame field (4.16) is along the focal submanifold $\mathbf{x} : U \subset M \to M_+ = f^{-1}\{1\} \subset S^n$
then the tube (3.1) with \( t = \pi/2 \) shows that the image of \( \bar{x} = e_0 : U \rightarrow M_+ \) is the same focal submanifold. If we let \( \bar{e}_0 = \bar{x} \), then by (4.15)

\[
\begin{align*}
\text{d}(x) &= \omega^\alpha e_\alpha - \omega^\alpha e_\alpha + \omega^\mu e_\mu \\
\text{d}(e_0) &= \omega^\alpha e_\alpha + \omega^\mu e_\mu
\end{align*}
\]

which shows that \( e_\alpha, e_\alpha, e_\mu \) are tangent to \( M_+ \) at \( \bar{x} = e_0 \), while \( \bar{e}_0, e_\mu \) are normal to \( M_+ \) at \( \bar{x} \). The second fundamental form at \( \bar{x} \) with respect to \( e_0 \) is

\[
\II_{e_0} e_0 = \text{d}(e_0) \cdot \text{d}(x) = \omega^\alpha \omega^\alpha - \sum \omega^\mu \omega^\mu
\]

which implies that \( V_+ \) is the +1 eigenspace and \( V_- \) is the −1 eigenspace of \( \II_{e_0} \) at \( \bar{x} \). Therefore, the principal curvature spaces of \( e_0 \) at \( x \) are

\[
\begin{align*}
V_+ &= V_+, \\
V_- &= V_-, \\
\tilde{V}_0 &= \text{span}\{e_a\}
\end{align*}
\]

It follows that a second order Darboux frame field along \( \bar{x} \) on \( U \) is

\[
\begin{align*}
\bar{x} &= e_0, \\
\bar{e}_0 &= \mathbf{x}, \\
\bar{e}_a &= e_{a+m}, \\
\bar{e}_{a+m} &= e_{a}, \\
\bar{e}_\alpha &= e_\alpha, \\
\bar{e}_\mu &= e_\mu
\end{align*}
\]

From (4.21) we see that

\[
\begin{align*}
\bar{\omega}^\alpha &= \omega^{a+m}, \\
\bar{\omega}^{a+m} &= \omega^\alpha, \\
\bar{\omega}^\alpha &= -\omega^\alpha, \\
\bar{\omega}^\mu &= \omega^\mu
\end{align*}
\]

is the coframe field dual to (4.23).

Of the forms in (4.18) for the frame field (4.23) and its coframe field (4.24), we consider

\[
\begin{align*}
d\bar{e}_\alpha \cdot e_\mu &= \bar{\beta}_\alpha^\mu = \bar{F}_\alpha^\mu \bar{\omega}^\alpha + \bar{F}_{\alpha a+m}^\mu \bar{\omega}^{a+m} \\
d\alpha \cdot e_\mu &= \beta_\alpha^\mu = F_\alpha^\mu \omega^\alpha + F_{\alpha a+m}^\mu \omega^{a+m}
\end{align*}
\]

to conclude that

\[
\begin{align*}
F_\alpha^\mu &= F_{\alpha a+m}^\mu, \\
F_{\alpha a+m}^\mu &= F_\alpha^\mu
\end{align*}
\]

Therefore, if \( v = \sum (x_\alpha e_\alpha + y_\mu e_\mu) \in V_+ \oplus V_- \), then

\[
\begin{align*}
\bar{p}_a(v) &= \sum_{\alpha, \mu} \bar{F}_\alpha^\mu x_\alpha y_\mu = \sum_{\alpha, \mu} F_{\alpha a+m}^\mu x_\alpha y_\mu = p_{a+m}(v)
\end{align*}
\]

where the polynomials \( \bar{p}_a \) and \( p_{a+m} \) are defined by these equations.

5. Consequences of the structure equations

We continue working with a second order frame field (4.16) along the focal submanifold \( \mathbf{x} \) defined in (3.1) with \( t = s_1 \). Equations (4.19) show that differentiating equations (2.9) is equivalent to differentiating
equations (4.18), which we now proceed to do. In preparation for this we first take the exterior differential of the coframe field (4.13) to obtain

\[
\begin{align*}
\omega^\alpha &= -\theta^\alpha_b \wedge \omega^b - F^\alpha_{pa} \omega^p \wedge \omega^a - F^\mu_{pa} \omega^p \wedge \omega^\mu - 4F^\mu_{\alpha a} \omega^\alpha \wedge \omega^\mu \\
\omega^b &= -\theta^b_q \wedge \omega^q + F^\alpha_{pa} \omega^a \wedge \omega^\alpha + F^\mu_{qa} \omega^a \wedge \omega^\mu + 4F^\mu_{\alpha p} \omega^\alpha \wedge \omega^\mu \\
\omega^a &= -\theta^a_\beta \wedge \omega^\beta - F^\alpha_{pa} \omega^a \wedge \omega^p + F^\mu_{aa} \omega^a \wedge \omega^\mu - F^\mu_{\alpha p} \omega^\alpha \wedge \omega^\mu \\
\omega^\mu &= -\theta^\mu_\nu \wedge \omega^\nu - F^\mu_{pa} \omega^a \wedge \omega^p - F^\mu_{aa} \omega^a \wedge \omega^\alpha + F^\mu_{\alpha p} \omega^\alpha \wedge \omega^\mu.
\end{align*}
\]  

(5.1)

We define the covariant derivatives of the tensors \( F^\alpha_{pa}, F^\mu_{pa}, F^\mu_{\alpha p} \) and \( F^\mu_{ap} \), respectively, to be the 1-forms

\[
\begin{align*}
F^\alpha_{pa} \omega^i &= dF^\alpha_{pa} - F^\alpha_{qa} \theta^q_a - F^\alpha_{pb} \theta^b_a + F^\beta_{pa} \theta^\alpha_a \\
F^\mu_{pa} \omega^i &= dF^\mu_{pa} - F^\mu_{qa} \theta^q_a - F^\mu_{pb} \theta^b_a + F^\nu_{pa} \theta^\mu_a \\
F^\mu_{aa} \omega^i &= dF^\mu_{aa} - F^\mu_{a\beta} \theta^a_\beta - F^\mu_{ab} \theta^b_a + F^\nu_{aa} \theta^\mu_a \\
F^\mu_{ap} \omega^i &= dF^\mu_{ap} - F^\mu_{a\beta} \theta^a_\beta - F^\mu_{ab} \theta^b_a + F^\nu_{ap} \theta^\mu_a
\end{align*}
\]  

(5.2)

Any other second order frame field along \( \mathbf{x} \) is given in terms of (4.15) by

\[
\mathbf{x}, e_0, \hat{e}_a, \hat{e}_p, \hat{e}_\alpha, \hat{e}_\mu
\]

where

\[
\hat{e}_a = A^b_a e_b, \quad \hat{e}_p = A^a_p e_q, \quad \hat{e}_\alpha = A^p_\alpha e_q, \quad \hat{e}_\mu = A^q_\mu e_q
\]

with \((A^b_a), (A^a_q) : U \to O(m)\) and \((A^p_\alpha), (A^q_\mu) : U \to O(N)\) smooth maps. If the coefficients with respect to this new frame field are denoted by the same letters covered by a hat, then the transformation rules are tensorial. For example,

\[
\hat{F}^\alpha_{pa} = A^\alpha_\beta F^\beta_{qa} A^q_p A^a_b, \quad \hat{F}^\mu_{pa} = A^\alpha_\beta F^\beta_{qa} A^q_p A^\mu_a A^a_b
\]

(5.3)

and so forth. If we take the exterior differential of the equations (4.18) and use (5.1) and (5.2) together with the Maurer-Cartan structure equations (2.4) we obtain the following sets of equations (compare [25, I, p. 536 and II, p. 45]).

\[
\begin{align*}
F^\alpha_{pa} F^\alpha_{qb} + F^\alpha_{pb} F^\alpha_{qa} - (F^\mu_{pa} F^\mu_{qb} + F^\mu_{pb} F^\mu_{qa}) &= 0 \\
F^\alpha_{pa} F^\beta_{pb} + F^\alpha_{pb} F^\beta_{pa} + 2(F^\alpha_{aa} F^\mu_{pb} + F^\mu_{ab} F^\alpha_{\beta a}) &= \delta_{a\beta} \delta_{ab} \\
F^\alpha_{pa} F^\beta_{qa} + F^\alpha_{qa} F^\beta_{pa} + 2(F^\mu_{aa} F^\mu_{\beta a} + F^\mu_{aq} F^\mu_{\beta p}) &= \delta_{pq} \delta_{a\beta} \\
F^\mu_{pa} F^\nu_{pb} + F^\mu_{pb} F^\nu_{pa} + 2(F^\mu_{aa} F^\nu_{pb} + F^\mu_{aq} F^\nu_{\beta p}) &= \delta_{ab} \delta_{\mu\nu} \\
F^\mu_{pa} F^\nu_{qa} + F^\mu_{qa} F^\nu_{pa} + 2(F^\mu_{aa} F^\nu_{pb} + F^\mu_{aq} F^\nu_{\beta p}) &= \delta_{pq} \delta_{\mu\nu} \\
F^\mu_{aa} F^\nu_{\beta a} + F^\mu_{\beta a} F^\nu_{aa} - (F^\mu_{\alpha p} F^\nu_{\beta p} + F^\mu_{\beta p} F^\nu_{\alpha p}) &= 0
\end{align*}
\]  

(5.4)
\[
F_{pab}^\alpha = -F_{p^a}^\mu F_{\mu}^{\alpha b} - 2F_{p^b}^\mu F_{\mu}^{\alpha a}
\]

(5.7)

\[
F_{paq}^\alpha = F_{pa}^\mu F_{\mu}^{\alpha q} + 2F_{op}^\mu F_{\mu}^{\alpha a}
\]

\[
F_{pa\beta} = 2F_{op}^\mu F_{\mu}^{\alpha \beta} - 2F_{p\beta}^\mu F_{\mu}^{\alpha a}
\]

\[
F_{pa\alpha}^\mu = -\frac{1}{2}F_{p^a}^\mu F_{\mu}^{\alpha b} + \frac{1}{2}F_{p^b}^\mu F_{\mu}^{\alpha a}
\]

(5.8)

\[
F_{p\alpha\beta} = \frac{1}{2}F_{p\alpha}^\mu F_{\mu}^{\beta a} + 2F_{p\alpha}^\mu F_{\mu}^{\beta a}
\]

\[
F_{p\alpha\nu} = \frac{1}{2}F_{p\nu}^\mu F_{\mu}^{\alpha a} + 2F_{p\nu}^\mu F_{\mu}^{\alpha a}
\]

(5.9)

\[
F_{p\alpha\beta} = \frac{1}{2}F_{p\alpha}^\mu F_{\mu}^{\beta a} - \frac{1}{2}F_{p\beta}^\mu F_{\mu}^{\alpha a}
\]

\[
F_{p\alpha\nu} = \frac{1}{2}F_{p\nu}^\mu F_{\mu}^{\alpha a} - \frac{1}{2}F_{p\nu}^\mu F_{\mu}^{\alpha a}
\]

(5.10)

\[
F_{p\alpha\mu} = -F_{p\alpha\mu} = -2F_{p\alpha}^\mu = -2F_{p\alpha}^\mu
\]

(5.11)

6. Second fundamental forms of a focal submanifold

Consider the focal submanifold \( x \) of (3.1) with \( t = s_1 \) with a second order frame field (4.16) along it on \( U \). For each point of \( x \), Corollary 6 tells us the principal curvatures of the second fundamental forms \( II_{e_a} \) of \( x \). In order to derive the consequence of this knowledge, we begin by finding the expression of \( II_{e_a} \) in terms of the orthonormal coframe field \( \omega^\mu, \omega^\alpha, \omega^\mu \) and from that obtain the matrices of the corresponding shape operators with respect to the orthonormal tangent frame field \( e_p, e_\alpha, e_\mu \). For our frame, equations (2.3) have become, in part,

\[
dx = \omega^a e_p + \omega^\alpha e_\alpha + \omega^\mu e_\mu
\]

(6.1)

\[
de_\alpha = (k_1 e_0 - x)\theta^a + \theta^b e_\alpha + \theta^\mu e_\mu + \theta^\alpha e_\alpha + \theta^\mu e_\mu
\]

The shape operator \( S_a \) is the symmetric operator on the tangent space at \( x \) given by

\[
II_{e_a} = -de_\alpha \cdot dx = dx \circ S_a \cdot dx
\]

(6.2)

That is, \( S_a \) is the tangential component of \(-de_\alpha \). Combining the second equation in (6.1) with (4.18), we find

\[
S_a = (2F_{\alpha a}^\mu e_\mu - F_{pa}^\alpha e_p)\omega^\alpha + (2F_{\alpha a}^\mu e_\alpha + F_{pa}^\mu e_p)\omega^\mu + (-F_{pa}^\alpha e_\alpha + F_{pa}^\mu e_\mu)\omega^p
\]
Recall the curvature spaces $V_0, V_+, V_-$ defined in (4.17). Define linear operators

\begin{align}
A_a &= 2F_{\alpha a}^\mu e_\alpha \omega^\mu : V_- \rightarrow V_+ \\
B_a &= -F^\alpha_{pa} e_\alpha \omega^p : V_0 \rightarrow V_+ \\
C_a &= F^\mu_{pa} e_\mu \omega^p : V_0 \rightarrow V_-
\end{align}

and their transposes

\begin{align}
^tA_a &= 2F_{\alpha a}^\mu e_\mu \omega^\alpha : V_+ \rightarrow V_- \\
^tB_a &= -F^\alpha_{pa} e_\mu \omega^\alpha : V_+ \rightarrow V_0 \\
^tC_a &= F^\mu_{pa} e_\mu \omega^\mu : V_- \rightarrow V_0
\end{align}

With respect to the orthogonal direct sum decomposition $V_+ \oplus V_- \oplus V_0$ of the tangent space to $\mathbf{x}$ at the point, the operator $S_a$ has the block form

\begin{equation}
S_a = \begin{pmatrix}
0 & A_a & B_a \\
^tA_a & 0 & C_a \\
^tB_a & ^tC_a & 0
\end{pmatrix}
\end{equation}

Restriction of the second fundamental forms $II_{e_0}$ and $II_{e_a}$ to $V_+ \oplus V_-$ defines quadratic forms

\begin{align}
p_a(x, y) &= II_{e_a}((x, y), (x, y)) = \sum_{\alpha} x_{\alpha}^2 - \sum_{\mu} y_{\mu}^2 \\
p_a(x, y) &= \frac{1}{4} II_{e_a}((x, y), (x, y)) = F_{\alpha \alpha}^\mu x_\alpha y_\mu
\end{align}

where $x = x_\alpha e_\alpha \in V_+$ and $y = y_\mu e_\mu \in V_-$. Note that by Corollary 6, the minimal polynomial of $S_a$ is $x(x^2 - 1)$, and therefore

\begin{equation}
S_a = S^3_a
\end{equation}

for all $a$ at every point of $U$.

**Proposition 7.** If $m < N$, then the operators $A_a$ in (6.3) must be linearly independent at every point of $U$.

**Proof.** Suppose that the operators $A_a$ are linearly dependent at a point $p \in U$. This means that there exists a unit vector $u = (u^a) \in \mathbb{R}^m$ such that

\begin{equation}
u^a F^\mu_{\alpha a} = 0
\end{equation}

for all $\mu$ and $\alpha$, at the point $p$. Then multiplying the second equation in (5.6) by $u^a u^b$, summing on $a$ and $b$ and using (6.8) gives

\begin{equation}2F^\alpha_{pa} u^a F^\beta_{pb} u^b = \delta_{\alpha\beta}
\end{equation}
Therefore,
\[
\{ \sqrt{2} \sum_{a,p} F_{pa}^{\alpha} u^a e_p : \alpha = 2m + 1, 2m + 2, \ldots, 2m + N \}
\]
is an orthonormal set of \( N \) vectors in the \( m \)-dimensional subspace \( V_0 \) defined in (4.17), which contradicts the assumption that \( m < N \). \( \Box \)

We need a condition which is stronger than the linear independence of the \( A_a \).

**Definition 8 (Spanning Property).** The focal submanifold \( \mathbf{x} \) satisfies the spanning property at a point of \( M \) if

(a). There exists a vector \( X = \sum_a x_a e_a \in V_+ \) such that the set of vectors \( \{ \sum_{a,\mu} F_{aa}^\mu x_a e_\mu : a = 1, \ldots, m \} \) in \( V_+ \) are linearly independent; and

(b). There exists a vector \( Y = \sum_\mu y_\mu e_\mu \in V_- \) such that the set of vectors \( \{ \sum_{a,\mu} F_{aa}^\mu y_\mu e_a : a = 1, \ldots, m \} \) in \( V_+ \) are linearly independent.

**Remark 9.** Condition (a) is equivalent to

(a'). There exists \( X = \sum_a x_a e_a \in V_+ \) such that \( \{ \sum_{a,\alpha,\mu} F_{aa}^\mu x_a y_\mu e_a : Y = y_\mu e_\mu \in V_- \} = \text{span} \{ e_1, \ldots, e_m \} \).

and (b) is equivalent to

(b'). There exists \( Y = \sum_\mu y_\mu e_\mu \in V_- \) such that \( \{ \sum_{a,\alpha,\mu} F_{aa}^\mu x_a y_\mu e_a : X = x_a e_a \in V_+ \} = \text{span} \{ e_1, \ldots, e_m \} \).

**Remark 10.** If \( \mathbf{x} \) satisfies the spanning property at a point of \( M \), then it satisfies it on some open neighborhood of the point by a standard argument on the rank of the \( N \times m \) matrix \( (F_{aa}^\mu x_a) \).

Let \( \mathbf{x}, e_0, e_\alpha, e_\mu, e_p, e_a, e_\mu \) be a second order frame field (4.16) along \( \mathbf{x} \) on \( U \), where \( \mathbf{x}(U) \subset M_+ \) is a focal submanifold. Let the same letters with bars denote the second order frame field (4.23) along \( \mathbf{x} = e_0 \) on \( U \). At each point of \( U \) define bihomogeneous polynomials \( p_a \) and \( \bar{p}_a \) in \( \mathbb{R}[x_\alpha, y_\mu] \) by

\[
(6.9) \quad p_a(x, y) = \sum_{a,\mu} F_{aa}^\mu x_\alpha y_\mu, \quad \bar{p}_a(x, y) = \sum_{a,\mu} \bar{F}_{aa}^\mu x_\alpha y_\mu
\]

where \( F_{aa}^\mu \) and \( \bar{F}_{aa}^\mu \) are defined in (4.18) for the respective frame fields.

**Proposition 11.** If at each point of \( U \) there exist polynomials \( f_{ab} \) in the polynomial ring \( \mathbb{R}[x_\alpha, y_\mu] \) such that

\[
(6.10) \quad \bar{p}_a = \sum_b f_{ab} p_b
\]
and if the spanning property holds for \( \mathbf{x} \) on \( U \), then there exists a second order frame field \( \mathbf{x}, e_0, \dot{e}_a, \ddot{e}_\alpha, \dddot{\varepsilon}_\mu \) along \( \mathbf{x} \) on \( U \) with respect to which

\[
\hat{F}_{\alpha a + m} = \hat{F}_{\alpha a}^\mu
\]

for all \( a, \alpha, \mu \), at each point of \( U \).

**Proof.** If we let \( p_a(x, y) = \sum_{\alpha, \mu} F_{\alpha a + m}^\mu x_\alpha y_\mu \), then by (4.26), \( p_{a+m} = \hat{p}_a \) and therefore (6.10) implies that at each point of \( U \)

\[
p_{a+m} = \sum_b f_{ab} p_b
\]

If we expand the right side of this equation in terms of the bihomogeneous components of the \( f_{ab} \) and collect all terms of the same bi-degrees, then all terms must cancel except those of bi-degree \((1, 1)\), since \( p_{a+m} \) has bi-degree \((1, 1)\). This results in an expression for \( p_{a+m} \) as a linear combination of the \( p_b \) with constant coefficients, since each \( p_b \) has bi-degree \((1, 1)\). Hence, we may assume that the \( f_{ab} \) in (6.12) are constant polynomials. Now (6.12) implies that

\[
F_{\alpha a + m}^\mu = \sum_b f_{ab} F_{\alpha b}^\mu
\]

for all \( \alpha, \mu \) at each point of \( U \). We claim that the functions \( f_{ab} : U \to \mathbb{R} \) are smooth. In fact, if we let \( A_{a+m} = 2 \sum_{\alpha, \mu} F_{\alpha a + m}^\mu e_\alpha \omega^\mu : V_- \to V_+ \) and let \( A_a \) be the operators defined in (6.3), then (6.13) implies that \( A_{a+m} = \sum_b f_{ab} A_b \). The spanning property implies that the operators \( A_b \) are linearly independent in \( \text{End}(V_-, V_+) \), and therefore at each point of \( U \) an inner product can be defined on this space of endomorphisms, depending smoothly on the point of \( U \), such that \( \{ A_b \} \) is an orthonormal set. Then \( f_{ab} = \langle A_{a+m}, A_b \rangle : U \to \mathbb{R} \) is smooth.

Fix \( \alpha = \alpha_0 \) and for each \( \mu \) define vectors in \( \mathbb{R}^m \)

\[
W_\mu = \langle F_{\alpha_0 1}^\mu, \ldots, F_{\alpha_0 m}^\mu \rangle, \quad V_\mu = \langle F_{\alpha_0 m+1}^\mu, \ldots, F_{\alpha_0 m+m}^\mu \rangle
\]

If we define the \( m \times m \) matrix \( B = (f_{ab}) \), then by (6.13), we have

\[
V_\mu = BW_\mu
\]

for each \( \mu \). The sixth equation in (5.6) says that for any \( \mu \) and \( \nu \)

\[
V_\mu \cdot V_\nu = W_\mu \cdot W_\nu
\]

Combining these equations, we have

\[
W_\mu \cdot W_\nu = BW_\mu \cdot BW_\nu
\]

for all \( \mu, \nu \). It follows that \( B \) is orthogonal, provided that the set \( \{ W_\mu \} \) spans \( \mathbb{R}^m \). By the spanning property, this is true for some choice of
α₀, for some choice of frame field. Therefore, assuming we have made that choice, we have a smooth map

\[ B = (f_{ab}) : U \to O(m) \]

Alter the second order frame field along \( x \) by

\[ \hat{e}_{a+m} = \sum_b e_{b+m} f_{ba} \]

leaving the other vectors in the frame unchanged. If we let \( \hat{F}^\mu_{\alpha a} \), etc. be the coefficients with respect to this new frame field, then by (5.5), we have \( \hat{F}^\mu_{\alpha a} = F^\mu_{\alpha a} \) and, also using (6.13), we have

\[ \hat{F}^\mu_{\alpha a+m} = \sum_b F^\mu_{\alpha b+m} f_{ba} = \sum_{b,c} f_{bc} F^\mu_{\alpha c} f_{ba} = \sum_c \hat{F}^\mu_{\alpha c} \delta_{ca} = \hat{F}^\mu_{\alpha a} \]

which proves (6.11).  

7. The Ferus-Karcher-Münzner Construction

Let \( P_0, P_1, \ldots, P_m \) be a Clifford system on \( \mathbb{R}^{2l} \). Recall that this means that these are symmetric operators on \( \mathbb{R}^{2l} \) satisfying

\[ P_i P_j + P_j P_i = 2\delta_{ij} I, \quad i, j = 0, 1, \ldots, m \]

It follows that each operator \( P_i \) is also orthogonal. For this section we modify the index conventions (4.6) by

\[ i, j, k \in \{0, \ldots, m\} \]

and now \( N = l - m - 1 \) and \( n + 1 = 2l \). If \( A = (A^i_j) \in SO(m+1) \), and if we let

\[ Q_i = A^i_j P_j \]

then \( Q_0, Q_1, \ldots, Q_m \) is also a Clifford system on \( \mathbb{R}^{2l} \). Since \( Q_0^2 = I \), the eigenvalues of \( Q_0 \) must be \( \pm 1 \). If \( E_{±} \) are the eigenspaces of \( Q_0 \), then \( \mathbb{R}^{2l} = E_{+} \oplus E_{-} \) is an orthogonal direct sum and \( E_{±} \) each has dimension \( l \), because for any \( a \), the operator \( Q_a \) interchanges \( E_{±} \).

Because \( P_0, \ldots, P_m \) are linearly independent,

\[ M_+ = \{ x \in S^{2l-1} \subset \mathbb{R}^{2l} : P_i x \cdot x = 0, \quad i = 0, \ldots, m \} \]

is a submanifold of \( S^{2l-1} \) of codimension \( m + 1 \). If \( x \in M_+ \), then \( Q_0 x, \ldots, Q_m x \) is an orthonormal set of unit normal vectors to \( M_+ \) in \( S^{2l-1} \). Therefore, this is a global frame field for the normal bundle of \( M_+ \) and the unit normal bundle of \( M_+ \) is isomorphic to the trivial bundle

\[ M = M_+ \times S^m \]
Consider the principal bundle

\[ SO(m+1) \to S^m \]
\[ A \mapsto A_0 \]

where for any \( A \in SO(m+1) \) we let \( A_i \) denote the \( i \)th column of \( A \). For a section \( A \) of (7.6), denote its pull-back to \( S^m \) of the Maurer-Cartan form of \( SO(m+1) \) by

\[ A^{-1}dA = \tau = (\tau^i_j) \]

an \( \mathfrak{o}(m+1) \)-valued form on \( S^m \). Then \( dA_i = A_j \tau^i_j \), and thus, for the Clifford systems

\[ Q_i = A^i_j P_j \]

depending on \( A \in SO(m+1) \), we have

\[ dQ_i = Q_j \tau^i_j \]

for each \( i \). Observe that \( \tau_0^1, \ldots, \tau_0^m \) is a local coframe field in \( S^m \). For each \((x, A_0) \in M = M_+ \times S^m\), there is an orthogonal direct sum

\[ \mathbb{R}^{2l} = \text{span}\{x\} \oplus M_+^+(x) \oplus T_0(x, A_0) \oplus T_+(x, A_0) \oplus T_-(x, A_0), \]

which is determined by the second fundamental form of \( M_+ \) (see Section 4.5 of [13, p. 488]). In Lemmas 12 - 14 below, we provide the details of the relationship between this decomposition and the second fundamental form of \( M_+ \). The subspaces of the decomposition are

\[ M_+^+(x) = \text{span}\{Q_0 x, \ldots, Q_m x\} = \text{span}\{P_0 x, \ldots, P_m x\} \]
\[ T_0(x, A_0) = \text{span}\{Q_a Q_0 x : \text{ for all } a\} \]
\[ T_+(x, A_0) = E_- \cap T_x M_+ = \{X \in E_-, X \cdot Q_i x = 0 \text{ for all } i\} \]
\[ = \{X \in \mathbb{R}^{2l} : Q_0 X = -X \text{ and } X \cdot P_i x = 0, \text{ for all } i\} \]
\[ T_-(x, A_0) = E_+ \cap T_x M_+ = \{X \in E_+, X \cdot Q_i x = 0 \text{ for all } i\} \]
\[ = \{X \in \mathbb{R}^{2l} : Q_0 X = X \text{ and } X \cdot P_i x = 0, \text{ for all } i\} \]

Then \( \dim M_+^+(x) = m + 1, \dim T_0(x, A_0) = m, \dim T_+(x, A_0) = N \) and \( \dim T_-(x, A_0) = N, \) where \( N = l - (m + 1) \). Notice that

\[ Q_0 : T_0(x, A_0) \to M_+^+(x) \]

because \( Q_0 Q_a Q_0 x = -Q_a x \in M_+^+, \) for any \( a \).

For any point in \( M = M_+ \times S^m \), there is an open neighborhood about it of the form \( U \times V \), where \( U \subset M_+ \) and \( V \subset S^m \), such that the section \( A \) of (7.6) is defined on \( V \) and such that there exist smooth
orthonormal bases $e_\alpha$ of $T_+(x, A_0)$ and $e_\mu$ of $T_-(x, A_0)$ on $U \times V$. This means that at each point of $U \times V$
\begin{align}
Q_0 e_\alpha &= -e_\alpha \quad \text{and} \quad e_\alpha \cdot Q_i x = 0 \\
Q_0 e_\mu &= e_\mu \quad \text{and} \quad e_\mu \cdot Q_i x = 0
\end{align}
(7.13)

Compose $x : M_+ \to S^{2l-1}$ with the projection $M = M_+ \times S^m \to M_+$ so that we may regard it as a mapping $x : M \to S^{2l-1}$. Then
\begin{align}
Q_0 e_\alpha &= e_\alpha \\
Q_0 e_\mu &= e_\mu
\end{align}
(7.14)

is a Darboux frame field along $x$ on $U \times V$, where the $e_i$ are normal vectors and the rest are tangent to $x$.

**Lemma 12.** For any $x \in M_+$
\begin{align}
Q_i Q_j Q_k x \cdot x &= 0
\end{align}
(7.15)

for all $i, j, k$ and
\begin{align}
L^a_{bc} &= Q_a Q_b Q_c e_0 \cdot x
\end{align}
(7.16)

is skew-symmetric in $a, b, c$.

**Proof.** If $i, j, k$ are distinct, then
\begin{align}
Q_i Q_j Q_k x \cdot x &= x \cdot Q_k Q_j Q_i x = -x \cdot Q_i Q_j Q_k x
\end{align}
which implies (7.15). If the indices are not distinct, then the product is a single $\pm Q_i$ and $Q_i x \cdot x = 0$ by definition of $M_+$.

If any two of $a, b, c$ are the same, then the product $Q_a Q_b Q_c$ is a single operator $\pm Q_a$, for some $a$, and we know that $Q_a e_0 \cdot x = 0$. If $a, b, c$ are distinct, then $Q_a Q_b Q_c$ changes sign if any two indices are switched. Therefore, $L^a_{bc}$ is skew-symmetric in $a, b, c$. \qed

**Lemma 13.** For the Darboux frame field (7.14) along $x$,
\begin{align}
Q_i x \cdot x &= 0, \quad \text{for all } i
\end{align}
(7.17)
\begin{align}
Q_i e_j \cdot e_k &= 0, \quad \text{for all } i, j, k
\end{align}
(7.18)
\begin{align}
Q_i e_p \cdot e_q &= 0, \quad \text{for all } i, p, q
\end{align}
(7.19)
\begin{align}
Q_a e_\alpha \cdot e_\beta &= 0, \quad \text{for all } a, \alpha, \beta
\end{align}
(7.20)
\begin{align}
Q_a e_\mu \cdot e_\nu &= 0, \quad \text{for all } a, \mu, \nu
\end{align}
(7.21)

at each point of $U \times V$.

**Proof.** The first equation follows from the definition of $M_+$. For the second equation
\begin{align}
Q_i e_j \cdot e_k &= Q_i Q_j x \cdot Q_k x = Q_k Q_i Q_j x \cdot x = 0
\end{align}
by Lemma 12. For the third equation
\[ Q_a e_p e_q = Q_a Q_{p-m} Q_0 x \cdot Q_{q-m} Q_0 x = -Q_{q-m} Q_a Q_{p-m} x \cdot x = 0 \]
by Lemma 12 and
\[ Q_0 e_p e_q = Q_0 Q_{p-m} Q_0 x \cdot Q_{q-m} Q_0 x = -x \cdot Q_{p-m} Q_{q-m} Q_0 x = 0 \]
by Lemma 12. Equations 4 and 5 follow from the observation made above that \( Q_a \) interchanges \( E_- \) and \( E_+ \). \( \square \)

**Lemma 14.** For the frame field (7.14),
\begin{align}
(7.22) & \quad d\mathbf{x} = \omega^p e_p + \omega^a e_a + \omega^\mu e_\mu \\
(7.23) & \quad d\mathbf{e}_0 = \omega^a e_a - \omega^\mu e_\mu + \omega^\alpha e_\alpha
\end{align}
where \( \omega^p, \omega^a, \omega^\mu \) are linearly independent one-forms on \( U \) with coefficients being functions on \( U \times V \), and
\begin{equation}
(7.24) \quad \omega^a = \tau_0^a - \omega^{a+m}
\end{equation}
A smooth coframe field on \( U \times V \) is given by \( \omega^a, \omega^p, \omega^\alpha, \omega^\mu \).

**Proof.** The expression (7.22) for \( d\mathbf{x} \) follows from the fact that \( \mathbf{x} : U \to \mathbb{R}^{2l} \) is an immersion and then \( \omega^A = d\mathbf{x} \cdot e_A \), for \( A = m + 1, \ldots, 2l - 1 \). Combining this with (7.9), we have
\begin{equation}
(7.25) \quad d\mathbf{e}_0 = dQ_0 \mathbf{x} + Q_0 d\mathbf{x}
\end{equation}
\begin{align*}
= \tau_0^a Q_a \mathbf{x} + \omega^{a+m} Q_0 Q_a Q_0 \mathbf{x} + \omega^a Q_0 e_a + \omega^\mu Q_0 e_\mu \\
= (\tau_0^a - \omega^{a+m}) e_a - \omega^a e_a + \omega^\mu e_\mu
\end{align*}
which proves (7.23). \( \square \)

For \( t \in \mathbb{R} \), the tube of radius \( t \) about \( M_+ \) is given by the immersion
\begin{equation}
(7.26) \quad \mathbf{x} : M \to S^{2l-1}, \quad \mathbf{x} = \cos t \mathbf{x} + \sin t \mathbf{e}_0
\end{equation}
A unit normal vector field along \( \mathbf{x} \) is
\begin{equation}
(7.27) \quad \mathbf{e}_0 = -\sin t \mathbf{x} + \cos t \mathbf{e}_0
\end{equation}
and a Darboux frame field along \( \mathbf{x} \) is given by
\begin{equation}
(7.28) \quad \mathbf{x}, e_a, e_p, e_\alpha, e_\mu, \mathbf{e}_0
\end{equation}
From (7.22) we compute
\begin{equation}
(7.29) \quad d\mathbf{x} = \sin t \omega^a e_a + \cos t \omega^p e_p \\
+ (\cos t - \sin t) \omega^a e_a + (\cos t + \sin t) \omega^\mu e_\mu
\end{equation}
\begin{equation}
\quad d\mathbf{e}_0 = \cos t \omega^a e_a - \sin t \omega^p e_p \\
- (\cos t + \sin t) \omega^a e_a + (\cos t - \sin t) \omega^\mu e_\mu
\end{equation}
which shows that
\begin{align}
\theta^a &= \sin t \omega^a, \quad \theta^p = \cos t \omega^p \\
\theta^\alpha &= (\cos t - \sin t)\omega^\alpha, \quad \theta^\mu = (\cos t + \sin t)\omega^\mu
\end{align}
(7.30)
is an orthonormal coframe field in $M$ for the metric $d\tilde{x} \cdot d\tilde{x}$ induced by $\tilde{x}$. The second fundamental form of $\tilde{x}$ is then
\begin{align}
II\tilde{e}_0 &= -d\tilde{x} \cdot d\tilde{e}_0 \\
&= -\cot t \theta^a \theta^a + \tan t \theta^p \theta^p + \frac{\cot t + 1}{\cot t - 1} \theta^\alpha \theta^\alpha - \frac{\cot t - 1}{\cot t + 1} \theta^\mu \theta^\mu
\end{align}
from which we conclude that the principal curvatures are the constants $\cot(-t)$ and $\cot(\pi/2 - t)$, each with multiplicity $m$ and the constants $\cot(\pi/4 - t)$ and $\cot(3\pi/4 - t)$, each with multiplicity $N$. In addition, the Darboux frame field (7.28) along $\tilde{x}$ is of second order. Therefore, the $\tilde{x}$ for $t \in \mathbb{R}$ is an isoparametric family of hypersurfaces in $S^{2l-1}$ and $\tilde{x}$ is a focal submanifold. This is the Ferus-Karcher-Münzner construction, (FKM construction) [13], of an isoparametric hypersurface from a given Clifford system.

We next calculate equations (4.18) for the FKM construction for a given Clifford system.

**Lemma 15.** For the Darboux frame field (7.14) along $\tilde{x}$, the coefficients of the forms $\theta^A_B = de_A \cdot e_B$ in (4.18) are given by
\begin{align}
F^\alpha_{pa} &= Q_{p-m}Q_a\tilde{x} \cdot e_\alpha, \quad F^\mu_{pa} = Q_{p-m}Q_a\tilde{x} \cdot e_\mu \\
F^\mu_{aa} &= -\frac{1}{2}Q_a e_\mu \cdot e_\alpha, \quad F^\mu_{ap} = -\frac{1}{2}Q_{p-m}e_\mu \cdot e_\alpha
\end{align}
(7.31)

**Proof.** These coefficients are determined by $\theta^p_a$, $\theta^a_p$ and $\theta^\alpha_p$. From (7.9) and (7.22) we have
\begin{align}
de_a &= dQ_a \tilde{x} + Q_a d\tilde{x} \\
&= -\tau_0^a e_0 + \tau_b^a e_{b+m} + \omega^b e_{b+m} + \omega^a Q_a e_\alpha + \omega^\mu Q_a e_\mu
\end{align}
(7.32)
and from (7.9) and (7.23) we have
\begin{align}
de_{a+m} &= dQ_a e_0 + Q_a de_0 \\
&= -\tau_0^a \tilde{x} + \tau_b^a e_{b+m} + \omega^b Q_a e_b - \omega^a Q_a e_\alpha + \omega^\mu Q_a e_\mu
\end{align}
(7.33)
Using Lemma 13 and (4.18) we have
\begin{align}
F^\alpha_{b+m} = F^\mu_{b+m} &= \theta^b_{a+m} = de_a \cdot e_{b+m} \\
&= \omega^a Q_a e_\alpha \cdot e_{b+m} + \omega^\mu Q_a e_\mu \cdot e_{b+m}
\end{align}
(7.34)
which implies that

\[
F_{b+m,a}^\alpha = Q_a e_\alpha \cdot e_{b+m} = Q_a e_\alpha \cdot Q_b Q_0 x
\]

\[
= Q_0 e_\alpha \cdot Q_a Q_b x = -e_\alpha \cdot Q_a Q_b x = Q_b Q_a x \cdot e_\alpha
\]

which is the first formula in (7.31), and similarly

\[
-F_{b+m,a}^\mu = Q_a e_\mu \cdot Q_b Q_0 x = Q_a e_\mu \cdot Q_b x = e_\mu \cdot Q_a Q_b x
\]

which gives the second formula in (7.31). In the same way,

(7.35)

\[
F_{b+m,a}^\alpha \omega^{b+m} - 2F_{aa}^\mu \omega^\mu = \theta^\alpha_a = d e_\alpha \cdot e_a
\]

\[
= \omega^{b+m} Q_a e_{b+m} \cdot e_\alpha + \omega^\mu Q_a e_\mu \cdot e_\alpha
\]

which implies that \(-2F_{aa}^\mu = Q_a e_\mu \cdot e_\alpha\), which is the third formula in (7.31). Next,

(7.36)

\[
F_{a+m,b}^\alpha \omega^b - 2F_{aa+m}^\mu \omega^\mu = \theta^\alpha_{a+m} = d e_{a+m} \cdot e_a
\]

\[
= \omega^b Q_a e_b \cdot e_\alpha + \omega^\mu Q_a e_\mu \cdot e_\alpha
\]

which implies that \(-2F_{aa+m}^\mu = Q_a e_\mu \cdot e_\alpha\), which is the fourth formula in (7.31).

\[\square\]

Corollary 16. With respect to a Darboux frame (7.14) along an FKM construction \(x : M \to S^{2l-1}\), the coefficients (7.31) satisfy the equations

(7.37)

\[
F_{aa+m}^\mu = F_{aa}^\mu
\]

\[
F_{a+m,b}^\alpha = -F_{b+m,a}^\alpha
\]

\[
F_{a+m,b}^\mu = -F_{b+m,a}^\mu
\]

Proof. From (7.31),

(7.38)

\[
F_{a+m,b}^\mu + F_{b+m,a}^\mu = (Q_a Q_b + Q_b Q_a) x \cdot e_\alpha = 0
\]

\[
F_{a+m,b}^\mu + F_{b+m,a}^\mu = (Q_a Q_b + Q_b Q_a) x \cdot e_\mu = 0
\]

\[\square\]

Proposition 17. For the Darboux frame field (7.14), at any point of \(U \times V \subset M\), the operators \(Q_0, Q_a\) are given by

(7.39)

\[
Q_0 x = e_0 \quad Q_0 e_0 = x \quad Q_0 e_\alpha = -e_{a+m}
\]

\[
Q_0 e_{a+m} = -e_a \quad Q_0 e_\alpha = -e_\alpha \quad Q_0 e_\mu = e_\mu
\]
and for each $a$

$$Q_a x = e_a$$
$$Q_a e_0 = e_{a+m}$$

(7.40)

$$Q_a e_b = \delta_{ab} x - L^c_{ab} e_{c+m} + F^\alpha_{a+m} e_a + F^\mu_{a+m} e_\mu$$
$$Q_a e_{b+m} = \delta_{ab} e_0 + \epsilon^c_{ab} + F^\alpha_{b+m} e_a - F^\mu_{b+m} e_\mu$$
$$Q_a e_a = F^\alpha_{a+m} e_b + F^\alpha_{b+m} e_a - 2F^\mu_{a+m} e_\mu$$
$$Q_a e_\mu = F^\mu_{a+m} e_b - 2F^\mu_{b+m} e_a$$

where the coefficients are defined in (7.16) and (7.31).

Proof. The expansion (7.39) of $Q_0$ can be verified by inspection. Also easy are the calculations $Q_a x = e_a$ and $Q_a e_0 = Q_a Q_0 x = e_{a+m}$. To calculate $Q_a$ on the remaining basis vectors, we use the fact that the basis is orthonormal. In the following calculations we use (7.1), (7.15), (7.14), (7.16) and (7.31).

$$Q_a e_b \cdot x = Q_a Q_b x \cdot x = \delta_{ab}$$
$$Q_a e_b \cdot e_0 = Q_a Q_b x \cdot Q_0 x = Q_0 Q_a Q_b x \cdot x = 0$$
$$Q_a e_b \cdot e_c = Q_a Q_b x \cdot Q_0 x = Q_0 Q_a Q_b x \cdot x = 0$$
$$Q_a e_b \cdot e_{c+m} = Q_a Q_b x \cdot Q_a Q_0 x = Q_b Q_a Q_c Q_0 x \cdot x = L^b_{ac} = -L^c_{ab}$$
$$Q_a e_b \cdot e_a = Q_a Q_b x \cdot e_a = F^\alpha_{a+m}$$
$$Q_a e_b \cdot e_\mu = Q_a Q_b x \cdot e_\mu = F^\mu_{a+m}$$

give the expansion of $Q_a e_b$.

$$Q_a e_{b+m} \cdot x = Q_a Q_b Q_0 x \cdot x = 0$$
$$Q_a e_{b+m} \cdot e_0 = Q_a Q_b Q_0 x \cdot Q_0 x = \delta_{ab}$$
$$Q_a e_{b+m} \cdot e_c = Q_a Q_b Q_0 x \cdot Q_0 x = Q_0 Q_a Q_b Q_0 x \cdot x = L^c_{ab}$$
$$Q_a e_{b+m} \cdot e_{c+m} = Q_a Q_b Q_0 x \cdot Q_0 x = -Q_0 Q_a Q_b x \cdot x = 0$$
$$Q_a e_{b+m} \cdot e_a = Q_a Q_b Q_0 x \cdot e_a = -Q_a Q_b x \cdot e_a = F^\alpha_{b+m}$$
$$Q_a e_{b+m} \cdot e_\mu = Q_a Q_b Q_0 x \cdot e_\mu = Q_a Q_b x \cdot e_\mu = -F^\mu_{b+m}$$

give the expansion of $Q_a e_{b+m}$. Using also (7.13), we find

$$Q_a e_a \cdot x = e_a \cdot Q_a x = 0$$
$$Q_a e_a \cdot e_0 = Q_a e_a \cdot Q_0 x = Q_a e_a \cdot x = 0$$
$$Q_a e_a \cdot e_b = Q_a e_a \cdot Q_b x = e_a \cdot Q_a Q_b x = F^\alpha_{a+m}$$
$$Q_a e_a \cdot e_\beta = 0$$
$$Q_a e_a \cdot e_\mu = Q_a e_a \cdot e_\mu = -2F^\mu_{a+a}$$
give the expansion of $Q_a e_\alpha$.

\[
Q_a e_\mu \cdot x = e_\mu \cdot Q_a x = 0 \\
Q_a e_\mu \cdot e_0 = Q_a e_\mu \cdot Q_0 x = -Q_a e_\mu \cdot x = 0 \\
Q_a e_\mu \cdot e_b = Q_a e_\mu \cdot Q_b x = e_\mu \cdot Q_a Q_b x = F^\mu_{a+m b} \\
Q_a e_\mu \cdot e_{b+m} = Q_a e_\mu \cdot Q_b Q_0 x = e_\mu \cdot Q_a Q_b x = -F^\mu_{b+m a} \\
Q_a e_\mu \cdot e_\alpha = -2 F^\mu_{a a} \\
Q_a e_\mu \cdot e_\nu = 0
\]

give the expansion of $Q_a e_\mu$. \hfill \Box

**Lemma 18.** For the Darboux frame field (7.14) along $x$,

\[
\begin{align*}
\theta^b_a &= \tau^b_a + L^b_{ac} \omega^{c+m} + F^a_{a+m b} \omega^a + F^\mu_{a+m b} \omega^\mu \\
\theta^b_{a+m} &= \tau^b_{a+m} + L^c_{ab} \omega^c + F^a_{a+m b} \omega^a + F^\mu_{a+m b} \omega^\mu 
\end{align*}
\]

and therefore

\[
\theta^b_a - \theta^b_{a+m} = L^b_{ac} (\omega^c + \omega^{c+m})
\]

**Proof.** Using (7.9) and (7.22), we find

\[
\begin{align*}
\theta^b_a &= de_a \cdot e_b = d(Q_a x) \cdot e_b \\
&= (\tau^b_a Q_x + \omega^p Q_a e_p + \omega^\alpha Q_a e_\alpha + \omega^\mu Q_a e_\mu) \cdot e_b \\
&= \tau^b_a + \omega^{c+m} Q_a e_{c+m} \cdot e_b + \omega^\alpha Q_a e_\alpha \cdot e_b + \omega^\mu Q_a e_\mu \cdot e_b
\end{align*}
\]

which combined with (7.40) gives the first formula in (7.41). The second formula is derived in the same way. \hfill \Box

8. **Necessary conditions to be FKM**

Let $\bar{x}, e_a, e_p, e_\alpha, e_\mu, \bar{e}_0$ be a second order frame field (2.2) in $U \subset M$ along an isoparametric hypersurface $\bar{x} : M \to S^n$. We continue using the index conventions in (4.6). Let $x = \cos s_1 \bar{x} + \sin s_1 \bar{e}_0$ be a focal submanifold and let $e_0 = -\sin s_1 \bar{x} + \cos s_1 e_0$ so that $x, e_0, e_a, e_p, e_\alpha, e_\mu$ is a Darboux frame field (4.16) along $x$ on $U$. Let $\omega^a, \omega^p, \omega^\alpha, \omega^\mu$ be its coframe field (4.13) on $U$. We look for conditions on this Darboux frame field which imply that $x$ comes from an FKM construction.

**Proposition 19.** Suppose that $x$ satisfies the spanning property (Definition 8) on $U$. If

\[
F^\mu_{a a+m} = F^\mu_{a a}
\]
on $U$, then
\begin{align}
F_{b+m\alpha}^\alpha + F_{a+m\beta}^\alpha &= 0 \\
F_{b+m\alpha}^\mu + F_{a+m\beta}^\mu &= 0 \\
\theta_b^\alpha - \theta_{b+m}^{\alpha+m} &= L_{bc}^\alpha(\omega^c + \omega^{c+m}), \text{ where } L_{bc}^\alpha = -L_{ac}^b = -L_{cb}^a
\end{align}
on $U$.

**Remark 20.** By Corollary 16 and Lemma 18, equations (8.1) - (8.4) hold for the Darboux frame field (7.14) defined along an FKM $x$.

**Proof.** The summation convention is not used in this proof. If we subtract the fourth equation in (5.2), with $p = a + m$, from the third equation in (5.2), we obtain
\begin{equation}
\sum_i (F_{\alpha\alpha i}^\mu - F_{\alpha a+m i}^\mu)\gamma^i = \sum_b F_{\alpha b}^\mu (\theta_{a+m}^{b+m} - \theta_{a}^b)
\end{equation}
Putting (8.1) into the second equation of (5.9) gives
\begin{equation}
F_{\alpha\alpha\beta}^\mu = \sum_b (F_{\alpha b}^\mu F_{b+m\alpha}^\beta + 2F_{\beta b}^\mu F_{a+m\alpha}^\alpha)
\end{equation}
and putting (8.1) into the second equation of (5.10) gives
\begin{equation}
F_{\alpha a+m\beta}^\mu = -\sum_b (F_{\alpha b}^\mu F_{a+m\beta}^\alpha + 2F_{\beta b}^\mu F_{a+m\alpha}^\alpha)
\end{equation}
Subtracting (8.7) from (8.6) we get
\begin{equation}
F_{\alpha\alpha\beta}^\mu - F_{\alpha a+m\beta}^\mu = \sum_b \left( F_{\alpha b}^\mu (F_{b+m\alpha}^\beta + F_{a+m\alpha}^\beta) + 2F_{\beta b}^\mu (F_{b+m\alpha}^\alpha + F_{a+m\alpha}^\alpha) \right)
\end{equation}
Likewise, using the third equation in (5.9) and in (5.10), gives
\begin{equation}
F_{\alpha\alpha\nu}^\mu - F_{\alpha a+m\nu}^\mu = \sum_b \left( F_{\alpha b}^\mu (F_{b+m\alpha}^{\nu} + F_{a+m\alpha}^{\nu}) + 2F_{\beta b}^\mu (F_{b+m\alpha}^\mu + F_{a+m\alpha}^\mu) \right)
\end{equation}
Expressing $\theta_{b}^\alpha - \theta_{b+m}^{\alpha+m}$ in terms of our coframe field, we have
\begin{equation}
\theta_{b}^\alpha - \theta_{b+m}^{\alpha+m} = \sum_c (L_{bc}^\alpha \omega^c + L_{bc+m}^\alpha \omega^{c+m}) + \sum_\alpha L_{\alpha b}^a \omega^\alpha + \sum_\mu L_{b\mu}^\alpha \omega^\mu
\end{equation}
where the coefficients are smooth functions on $U$, each skew-symmetric in $a, b$.

By the spanning property, as expressed in (a') of Remark 9, we may assume the basis of $V_+$ chosen so that for some $\alpha$, the set of vectors
\begin{equation}
\{ \sum_a F_{\alpha a}^\mu e_a : \text{ all } \mu \}
\end{equation}
spans $V_0$. Fix this choice of $\alpha$. Substitute (8.10) into (8.5) and compare the coefficients of $\omega^\alpha$ on each side to obtain

\begin{equation}
F^\mu_{\alpha a a} - F^\mu_{\alpha a + m a} = \sum_b F^\mu_{ab} L^a_{ba}
\end{equation}

Compare this to (8.8), in which we set $\beta = \alpha$, to obtain

\begin{equation}
\sum_b F^\mu_{ab} (3(F^\alpha_{b+a} + F^\alpha_{a+m b}) - L^a_{ba}) = 0
\end{equation}

for all $a$ and $\mu$. By the spanning property, then, the vectors

\[\sum_b (3(F^\alpha_{b+a} + F^\alpha_{a+m b}) - L^a_{ba}) e_b\]

for each $a$ and $\mu$, are orthogonal to every vector in $V_0$. Therefore,

\begin{equation}
3(F^\alpha_{b+a} + F^\alpha_{a+m b}) = L^a_{ba}
\end{equation}

The left side of this equation is symmetric in $a, b$, while the right side is skew-symmetric in $a, b$. Therefore, for our choice of $\alpha$, (8.2) holds and

\begin{equation}
L^a_{ba} = 0
\end{equation}

for all $a, b$. Now, (8.8) becomes, for our choice of $\alpha$ and for any $\beta$,

\begin{equation}
F^\mu_{\alpha a \beta} - F^\mu_{\alpha a + m \beta} = \sum_b F^\mu_{ab} (F^\beta_{b+a} + F^\beta_{a+m b})
\end{equation}

Substitute (8.10) into (8.5) and compare the coefficient of $\omega^\beta$ with (8.15) to obtain

\[\sum_b F^\mu_{ab} (F^\beta_{b+a} + F^\beta_{a+m b} - L^a_{b\beta}) = 0\]

for all $a, \beta$, and $\mu$. Again, the spanning property then implies that

\[F^\beta_{b+a} + F^\beta_{a+m b} = L^a_{b\beta}\]

for all $a, b$, and $\beta$. Hence, as before, each side of this equation must be zero. Therefore, (8.2) and (8.14) hold for all $a, b$, and $\alpha$.

We can prove (8.3) and

\begin{equation}
L^a_{b \mu} = 0
\end{equation}

for all $a, b$ and $\mu$ in a similar way, by first fixing an appropriate $\mu$ and comparing coefficients of $\omega^\mu$ in (8.5) after substitution of (8.10) into it. In this case (b') of the spanning property is used.

With (8.2) and (8.3) now true, we see that (8.8) and (8.9) become

\begin{equation}
F^\mu_{\alpha a \beta} = F^\mu_{\alpha a + m \beta}, \quad F^\mu_{\alpha a \nu} = F^\mu_{\alpha a + m \nu}
\end{equation}
and (8.14) and (8.16) substituted into (8.10) give

\[(8.18)\]
\[\theta^a_b - \theta^a_{b+m} = \sum_c (L^a_{bc} \omega^c + L^a_{bc+m} \omega^{c+m})\]

Substitute this into (8.5) and compare coefficients of \(\omega^c\) and \(\omega^{c+m}\) to get

\[(8.19)\]
\[\sum_b F^\mu_{ab} L^a_{bc} = F^\mu_{aac} - F^\mu_{a\alpha a+m c}\]
\[\sum_b F^\mu_{ab} L^a_{b c+m} = F^\mu_{aa c+m} - F^\mu_{a\alpha a+m c+m}\]

Subtracting gives

\[(8.20)\]
\[\sum_c F^\mu_{a\alpha c}(L^a_{cb} - L^a_{c b+m}) = F^\mu_{a\alpha a+b} - F^\mu_{a\alpha a+m b} - F^\mu_{a a b+m} + F^\mu_{a a a+m b+m}\]

We want to show now that the right hand side of this equation is zero on \(U\). To that end, we begin with the first equation in (5.9), which says

\[(8.21)\]
\[F^\mu_{a a b} = -\frac{1}{2} \sum_c F^\mu_{c+m a} F^\alpha_{c+m b} + \frac{1}{2} \sum_c F^\mu_{c+m b} F^\alpha_{c+m a}\]

and (5.11) says

\[(8.22)\]
\[F^\mu_{a a a+m b} = \frac{1}{2} F^\mu_{a a+m b a}, \quad F^\mu_{a a b+m} = \frac{1}{2} F^\mu_{b+m a a}\]

and the first equation in (5.10) says

\[(8.23)\]
\[F^\mu_{a a+m b+m} = \frac{1}{2} \sum_c F^\mu_{a a+m c} F^\alpha_{b+m c} - \frac{1}{2} \sum_c F^\mu_{b+m c} F^\alpha_{a+m c}\]

Hence, using (8.2) and (8.3), the right hand side of (8.20) is

\[
\begin{align*}
F^\mu_{a a b} - F^\mu_{a a+m b} - F^\mu_{a a b+m} + F^\mu_{a a+m b+m} &= \\
- \frac{1}{2} &\sum_c F^\mu_{c+m a} F^\alpha_{c+m b} + \frac{1}{2} \sum_c F^\mu_{c+m b} F^\alpha_{c+m a} - \frac{1}{2} F^\mu_{a a+m b a} - \frac{1}{2} F^\mu_{b+m a a} \\
+ \frac{1}{2} &\sum_c F^\mu_{a a+m c} F^\alpha_{b+m c} - \frac{1}{2} \sum_c F^\mu_{b+m c} F^\alpha_{a+m c} = -\frac{1}{2} (F^\mu_{a a+m b a} + F^\mu_{b+m a a}) \\
+ \frac{1}{2} &\sum_c F^\mu_{a a+m c} (F^\alpha_{c+m b} + F^\alpha_{b+m c}) + \frac{1}{2} \sum_c F^\mu_{c+m b} (F^\alpha_{c+m a} + F^\alpha_{a+m c}) \\
= &\ -\frac{1}{2} (F^\mu_{a a+m b a} + F^\mu_{b+m a a})
\end{align*}
\]
and so we want to show that this last term is zero on $U$ when (8.1), (8.2) and (8.3) hold. By the second equation in (5.2),

$$\sum_i F_{a+m}^\mu b \omega^i = dF_{a+m}^\mu b - \sum_c F_{c+m}^\mu b \theta_{a+m}^c - \sum_c F_{a+m}^\mu c \theta_b^c + \sum_{\nu} F_{a+m}^\nu b \theta_{\nu}^c$$

and

$$\sum_i F_{b+m}^\mu a \omega^i = dF_{b+m}^\mu a - \sum_a F_{c+m}^\mu a \theta_{b+m}^c - \sum_c F_{b+m}^\mu c \theta_a^c + \sum_{\nu} F_{b+m}^\nu a \theta_{\nu}^c$$

Sum these two equations and use (8.2) and (8.3) to get

$$\sum_i (F_{a+m}^\mu b + F_{b+m}^\mu a) \omega^i = \sum_c (F_{b+m}^\mu c (\theta_{a+m}^c - \theta_a^c) + F_{a+m}^\mu c (\theta_{b+m}^c - \theta_b^c))$$

By (8.18), the right hand side of this equation is in the span of the set of 1-forms $\{\omega^c, \omega^{c+m} : c = 1, \ldots, m\}$, and therefore the coefficients of $\omega^\alpha$ and $\omega^\mu$ on the left hand side must vanish, to give

$$F_{a+m}^\mu b a + F_{b+m}^\mu b a = 0, \quad F_{a+m}^\mu b a + F_{b+m}^\mu b a = 0$$

and we have finally proved that the right hand side of (8.20) is zero on $U$, and therefore

$$\sum_b F_{a+m b a} (L_{b c}^a - L_{b c+m}^a) = 0$$

on $U$, for all $a$, $c$, $\alpha$, and $\mu$. Multiplying this equation by the $X = \sum x_\alpha e_\alpha$ of (a) of the spanning property, we conclude that

$$L_{b c}^a - L_{b c+m}^a = 0$$

on $U$ for all $a$, $b$, $c$. Substitution of this into (8.18) gives

$$\theta_{b}^a - \theta_{b+m}^a = \sum_c L_{b c}^a (\omega^c + \omega^{c+m})$$

To complete the proof of (8.4), it remains to show that

$$L_{b c}^a + L_{c b}^a = 0$$

on $U$, for all $a$, $b$, $c$. By (5.2), (8.1) and (8.27), and the known skew-symmetry $L_{b c}^a = -L_{b c}^a$, we have

$$\sum_i F_{a+m i}^\mu \omega^i = \sum_i F_{a i}^\mu \omega^i + \sum_{b,c} F_{a b}^\mu L_{a c}^b (\omega^c + \omega^{c+m})$$

Comparing the coefficients of $\omega^c$, we have

$$F_{a+m c}^\mu = F_{a c}^\mu + \sum_b F_{a b}^\mu L_{b c}^a$$
Interchanging $a$ and $c$ and then summing, we have

\begin{equation}
F^\mu_{\alpha a + m c} + F^\mu_{\alpha c + m a} = F^\mu_{\alpha a c} + F^\mu_{\alpha c a} + \sum_b F^\mu_{ab} (L^b_{ac} + L^b_{ca})
\end{equation}

By the first equation in (5.9), we have

\begin{equation}
F^\mu_{\alpha a c} + F^\mu_{\alpha c a} = 0
\end{equation}

Hence

\begin{equation}
F^\mu_{\alpha a + m c} + F^\mu_{\alpha c + m a} = \sum_b F^\mu_{ab} (L^b_{ac} + L^b_{ca})
\end{equation}

on $U$ for all $\alpha$ and $\mu$. In (8.29) compare the coefficients of $\omega^{c+m}$ to get

\begin{equation}
F^\mu_{\alpha a + m c + m} = F^\mu_{\alpha a c + m} + \sum_b F^\mu_{ab} L^b_{ac}
\end{equation}

Interchange $a$ and $c$ and sum, to get

\begin{equation}
F^\mu_{\alpha a + m c + m} + F^\mu_{\alpha c + m a + m} = F^\mu_{\alpha a c + m} + F^\mu_{\alpha c a + m} + \sum_b F^\mu_{ab} (L^b_{ac} + L^b_{ca})
\end{equation}

By the first equation in (5.10),

\begin{equation}
F^\mu_{\alpha a + m c + m} + F^\mu_{\alpha c + m a + m} = 0
\end{equation}

and the last equation in (5.11) says that

\begin{equation}
F^\mu_{\alpha a + m c + m} = F^\mu_{\alpha a c + m} = F^\mu_{\alpha a + m c}
\end{equation}

Therefore, (8.34) is

\begin{equation}
F^\mu_{\alpha c + m a} + F^\mu_{\alpha a + m c} = -\sum_b F^\mu_{ab} (L^b_{ac} + L^b_{ca})
\end{equation}

Combining this with (8.33), we conclude that

\begin{equation}
\sum_b F^\mu_{ab} (L^b_{ac} + L^b_{ca}) = 0
\end{equation}

for all $a, c, \alpha, \mu$. The spanning property then implies (8.28). \hfill \Box

Resume use of the summation convention.

**Proposition 21.** If equations (8.1) through (8.4) hold on $U$, then

\begin{align}
(8.37) & \quad F^\alpha_{c+m,a} L^c_{bd} + F^\alpha_{c+m,b} L^c_{ad} = 2(F^\mu_{aa} F^\mu_{d+m,b} + F^\mu_{ab} F^\mu_{d+m,a}) \\
(8.38) & \quad F^\mu_{c+m,a} L^c_{bd} + F^\mu_{c+m,b} L^c_{ad} = 2(F^\alpha_{b+m} F^\alpha_{a+m,d} F^\mu_{ab} + F^\alpha_{b+m,d} F^\alpha_{a+m} F^\mu_{ab}) \\
(8.39) & \quad F^\mu_{\alpha b+m,a} = L^c_{ba} F^\mu_{ac} - \frac{1}{2} F^\mu_{d+m,b} F^\alpha_{d+m,a} + \frac{1}{2} F^\mu_{d+m,a} F^\alpha_{d+m,b}
\end{align}
Proof. These identities come from differentiating (8.1) through (8.3). Using our definition of covariant derivative in (5.2), we have

\[
\begin{align*}
\text{(8.40)} & \quad dF^\alpha_{b+m/a} + F^\beta_{b+m/a} \theta^\alpha_{\beta} - F^\alpha_{c+m/b} \theta^c_{b+m} - F^\alpha_{b+m/c} \theta^c_a = F^\alpha_{b+m/a} \omega^i \\
\text{(8.41)} & \quad dF^\alpha_{a+m/b} + F^\beta_{a+m/b} \theta^\alpha_{\beta} - F^\alpha_{c+m/a} \theta^c_{a+m} - F^\alpha_{a+m/c} \theta^c_b = F^\alpha_{a+m/b} \omega^i
\end{align*}
\]

Summing these two equations and using (8.2) and (8.4), we get

\[
\begin{align*}
\text{(8.42)} & \quad (F^\alpha_{c+m/a} L^c_{bd} + F^\alpha_{c+m/b} L^c_{ad})(\omega^d + \omega^{d+m}) = (F^\alpha_{b+m/a} + F^\alpha_{a+m/b}) \omega^i
\end{align*}
\]

Equating the coefficients of $\omega^d$, we have

\[
\begin{align*}
\text{(8.43)} & \quad F^\alpha_{c+m/a} L^c_{bd} + F^\alpha_{c+m/b} L^c_{ad} = F^\alpha_{b+m/\alpha d} + F^\alpha_{a+m/\beta d}
\end{align*}
\]

From (5.7) we see that the right side of (8.42) is

\[
\begin{align*}
\text{(8.44)} & \quad -F^\mu_{b+m/\alpha d} F^\mu_{\alpha d} - 2F^\mu_{b+m/\alpha d} F^\mu_{\alpha a} - F^\mu_{a+m/\alpha b} F^\mu_{\alpha a} - 2F^\mu_{a+m/\alpha d} F^\mu_{\alpha b} \\
& \quad = 2F^\mu_{d+m/\alpha b} F^\mu_{\alpha a} + 2F^\mu_{d+m/\alpha a} F^\mu_{\alpha b}
\end{align*}
\]

where the last equality comes from using (8.3). Now (8.37) follows from (8.42) and (8.43). Equating the coefficients of $\omega^{d+m}$ in (8.41) leads again to (8.37). Equating the other coefficients leads to the identities

\[
\begin{align*}
\text{(8.45)} & \quad F^\alpha_{b+m/\alpha a} + F^\alpha_{a+m/\beta b} = 0 \quad \text{and} \quad F^\alpha_{b+m/\alpha a} + F^\alpha_{b+m/\alpha a} = 0
\end{align*}
\]

We next find the consequences of taking the covariant derivative of equation (8.3). Again by (5.2), we have

\[
\begin{align*}
\text{(8.46)} & \quad (F^\mu_{c+m/a} L^c_{bd} + F^\mu_{c+m/b} L^c_{ad})(\omega^d + \omega^{d+m}) = (F^\mu_{b+m/a} + F^\mu_{a+m/b}) \omega^i
\end{align*}
\]

Equating the coefficients of $\omega^d$ we have

\[
\begin{align*}
\text{(8.47)} & \quad F^\mu_{c+m/a} L^c_{bd} + F^\mu_{c+m/b} L^c_{ad} = F^\mu_{b+m/\alpha d} + F^\mu_{a+m/\beta d}
\end{align*}
\]

By (5.8), the right side of (8.47) is

\[
\begin{align*}
\text{(8.48)} & \quad F^\mu_{b+m/a} F^\mu_{d/\alpha a} + 2F^\alpha_{b+m/\alpha d} F^\mu_{\alpha a} + F^\alpha_{a+m/\alpha b} F^\mu_{\alpha a} + 2F^\alpha_{a+m/\alpha d} F^\mu_{\alpha b}
\end{align*}
\]

Using (8.2) in (8.48), we then arrive at (8.38). Equating coefficients of $\omega^{d+m}$ in (8.46) also leads to (8.38). Equating coefficients of $\omega^\alpha$ of $\omega^\mu$ gives

\[
\begin{align*}
\text{(8.49)} & \quad F^\mu_{b+m/\alpha a} + F^\mu_{b+m/\alpha a} = 0 \quad \text{and} \quad F^\mu_{b+m/\alpha a} + F^\mu_{b+m/\alpha a} = 0
\end{align*}
\]

Finally, substitute the first equation of (5.9) into (8.30) to arrive at (8.39).
We define the covariant derivatives of the \( L^{a}_{bc} \) to be the coefficients \( L^{a}_{bci} \) of the 1-form
\[
(8.50) \quad dL^{a}_{bc} + L^{d}_{bc} \theta^{a}_{d} - L^{a}_{dc} \theta^{d}_{b} - L^{a}_{bd} \theta^{d}_{c} = L^{a}_{bci} \omega^{i}
\]

Remark 22. If the \( L^{a}_{bc} \) are skew-symmetric in all three indices, then the functions \( L^{a}_{bci} \) are skew-symmetric in \( a, b, c \).

Proposition 23. If equations (8.1) through (8.4) hold, then the \( L^{a}_{bcd} \) are skew-symmetric in all four indices, and
\[
(8.51) \quad L^{a}_{bcd} = \frac{1}{2} (\delta_{ad} \delta_{bc} - \delta_{ac} \delta_{bd}) + \frac{1}{2} (L^{e}_{ce} L^{e}_{bd} - L^{e}_{de} L^{e}_{bc})
\]
\[
+ F^{\alpha}_{c+m} F^{\alpha}_{d+m} - F^{\mu}_{c+m} F^{\mu}_{d+m}
\]
\[
(8.52) \quad L^{a}_{bci,d+m} = \frac{1}{2} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + L^{e}_{be} L^{e}_{dc} + \frac{1}{2} (L^{e}_{ce} L^{e}_{bd} - L^{e}_{de} L^{e}_{bc})
\]
\[
+ F^{\alpha}_{d+m} F^{\alpha}_{c+m} - F^{\mu}_{d+m} F^{\mu}_{c+m}
\]
\[
(8.53) \quad L^{a}_{bce} = L^{a}_{be} F^{\alpha}_{c+m} + 2(F^{\mu}_{aa} F^{\mu}_{c+m} - F^{\mu}_{ab} F^{\mu}_{c+m})
\]
\[
(8.54) \quad L^{a}_{bcm} = L^{a}_{be} F^{\alpha}_{c+m} + 2(F^{\mu}_{ab} F^{\alpha}_{c+m} - F^{\mu}_{ab} F^{\alpha}_{c+m})
\]
\[
(8.55) \quad 2\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} - \delta_{ab} \delta_{dc} =
\]
\[
L^{a}_{bci} dL^{c}_{de} + L^{a}_{de} L^{c}_{bc} + 2(F^{\alpha}_{b+m} F^{\alpha}_{d+m} a + F^{\alpha}_{b+m} F^{\alpha}_{d+m})
\]
\[
(8.56) \quad L^{a}_{bci,d+m} + L^{a}_{bd,c+m} = 0
\]

Proof. This proposition is a consequence of taking the exterior derivative of (8.4). Notice that (8.56) follows directly from (8.52).

Using (4.18) and the structure equations (2.4), we find
\[
d(\theta^{a}_{b} - \theta^{a+m}_{b+m}) = \omega^{a} \wedge \omega^{b} - \omega^{a+m} \wedge \omega^{b+m}
\]
\[
+ (F^{\alpha}_{c+m} F^{\alpha}_{d+m} a + F^{\mu}_{c+m} F^{\mu}_{d+m})(\omega^{a} \wedge \omega^{c+m} - \omega^{c} \wedge \omega^{d})
\]
\[
+ [L^{a}_{de} \theta^{a}_{d} - L^{a}_{de} \theta^{a}_{d} + L^{e}_{de} L^{e}_{be} (\omega^{d} + \omega^{d+m}) + 2(F^{\mu}_{aa} F^{\mu}_{c+m} - F^{\mu}_{ab} F^{\mu}_{c+m}) \omega^{a}
\]
\[
+ 2(F^{\mu}_{ab} F^{\alpha}_{c+m} - F^{\mu}_{ab} F^{\alpha}_{c+m})(\omega^{c} + \omega^{c+m})
\]
Using (5.1), we find
\[
d(L^{a}_{bc} (\omega^{c} + \omega^{c+m})) = (dL^{a}_{bc} - L^{a}_{bd} \theta^{c}_{d} - L^{a}_{be} L^{e}_{d} \omega^{d+m}
\]
\[
- L^{a}_{bd} F^{\alpha}_{d+m} \omega^{a} - L^{a}_{bd} F^{\mu}_{d+m} \omega^{\mu}) \wedge (\omega^{c} + \omega^{c+m})
\]
The exterior differential of (8.4) is obtained by equating the preceding two equations and using (8.50), to get

\[
L^a_{bec} \omega^i \wedge (\omega^e + \omega^{e+m}) = \\
[-\delta_{ae}\omega^b - \delta_{be}\omega^{a+m}] + (L^a_{dc}L^d_{be} + F^\alpha_{e+m,a}F^{\alpha}_{c+m,b} + F^{\mu}_{c+m,a}F^{\mu}_{e+m,b})\omega^c \\
+ (L^a_{bd}L^d_{ce} + F^\alpha_{e+m,a}F^{\alpha}_{c+m,b} + F^{\mu}_{c+m,a}F^{\mu}_{e+m,b})\omega^e + (L^a_{be}F^{\alpha}_{c+m,e} + 2F^{\alpha}_{e+m,a}F^{\alpha}_{e+m,b} - 2F^{\alpha}_{e+m,a}F^{\alpha}_{e+m,ab})\omega^\alpha \\
+ (L^a_{be}F^{\mu}_{c+m,e} + 2F^{\mu}_{e+m,a}F^{\mu}_{e+m,b} - 2F^{\mu}_{e+m,a}F^{\mu}_{e+m,ab})\omega^\mu \wedge (\omega^e + \omega^{e+m})
\]

Equating the skew-symmetrized coefficients of \(\omega^c \wedge \omega^e\) in this equation, we have

\[
L^a_{bec} - L^a_{bce} = L^a_{dc}L^d_{be} - L^a_{dc}L^d_{be} - \delta_{ae}\delta_{bc} + \delta_{ae}\delta_{be} \\
+ F^\alpha_{e+m,a}F^{\alpha}_{c+m,b} - F^\alpha_{c+m,a}F^{\alpha}_{e+m,b} \\
+ F^{\mu}_{e+m,a}F^{\mu}_{c+m,b} - F^{\mu}_{c+m,a}F^{\mu}_{e+m,b}
\]

Rewrite (8.58) with b and e interchanged and add the result to (8.58). Using that \(L^a_{be}, L^a_{bce}, F^\alpha_{e+m,b}\) and \(F^{\mu}_{e+m,b}\) are all skew-symmetric in b and e, we get from this sum

\[
L^a_{bec} + L^a_{ceb} = L^a_{dc}L^d_{be} + L^a_{ed}L^d_{bd} + \delta_{ae}\delta_{bc} + \delta_{ab}\delta_{ec} - 2\delta_{ac}\delta_{be} \\
+ F^\alpha_{b+m,a}F^{\alpha}_{e+m,c} + F^\alpha_{b+m,a}F^{\alpha}_{e+m,c} \\
+ F^{\mu}_{b+m,a}F^{\mu}_{e+m,c} + F^{\mu}_{b+m,a}F^{\mu}_{e+m,c}
\]

Equating the coefficients of \(\omega^c \wedge \omega^{e+m}\) in (8.57), we find

\[
L^a_{bec} - L^a_{bce+m} = L^a_{dc}L^d_{be} - L^a_{bd}L^d_{ec} - L^a_{dc}L^d_{bc}
\]

Rewrite this equation with b and c interchanged and add the result to (8.60). From the skew-symmetry of \(L^a_{be}\) and \(L^a_{bce}\) in a, b, c, it follows from this sum that

\[
L^a_{bec} + L^a_{ceb} = 0
\]

from which we conclude that \(L^a_{bce}\) is skew-symmetric in all four indices. Putting (8.61) into (8.59), interchanging \(d\) and \(e\) and using the first equation in (5.6), we arrive at (8.55). Putting (8.61) into (8.58) and using the first equation of (5.6), we get (8.51). Substitute (8.51) into (8.60) to obtain (8.52). Go back to (8.57) and equate coefficients of \(\omega^a \wedge \omega^c\) to obtain (8.53), and equate coefficients of \(\omega^\mu \wedge \omega^c\) to obtain (8.54).
9. A SUFFICIENT CONDITION TO BE FKM

Let \( \tilde{x}, e_a, e_p, e_\alpha, e_\mu, \tilde{e}_0 \) be a second order frame field (2.2) in \( U \subset M \) along an isoparametric hypersurface \( \tilde{x} : M \to S^n \subset \mathbb{R}^{n+1} \). We continue using the index conventions in (4.6). Let \( x = \cos s_1 \tilde{x} + \sin s_1 \tilde{e}_0 \) be a focal submanifold and let \( e_0 = -\sin s_1 \tilde{x} + \cos s_1 \tilde{e}_0 \) so that

\[
(9.1) \quad x, e_0, e_a, e_p, e_\alpha, e_\mu
\]

is a Darboux frame field (4.16) along \( x \) on \( U \). Let

\[
(9.2) \quad \omega^a, \omega^p, \omega^\alpha, \omega^\mu
\]

be its coframe field (4.13) on \( U \).

**Theorem 24.** If \( x \) satisfies the spanning property (Definition 8) and condition (8.1), \( F_\alpha^{\mu a + m} = F_\alpha^{\nu a} \), on \( U \), then it comes from an FKM construction.

**Proof.** It is sufficient to prove the theorem locally, on some open neighborhood, because isoparametric hypersurfaces are algebraic. For each point in \( U \), the vectors of our Darboux frame field (9.1) form an orthonormal basis of \( \mathbb{R}^{n+1} \). Linear operators \( Q_0, Q_a \) on \( \mathbb{R}^{n+1} \), depending on the point in \( U \), can thus be defined by (7.39) and (7.40), which we recopy here for easier reference

\[
(9.3) \quad Q_0 x = e_0 \quad Q_0 e_0 = x \quad Q_0 e_a = -e_{a+m} \\
Q_0 e_{a+m} = -e_a \quad Q_0 e_\alpha = -e_\alpha \quad Q_0 e_\mu = e_\mu
\]

and for each \( a \)

\[
(9.4) \quad Q_a x = e_a \\
Q_a e_0 = e_{a+m} \\
Q_a e_{a + m} = \delta_{ab} e_0 + L^c_{ab} e_c + F_\alpha^{a+m} e_\alpha - F_\beta^{b+a} e_\mu \\
Q_a e_\alpha = F_\alpha^{a+m} e_\beta + F_\beta^{b+a} e_{a+b+m} - 2F_\alpha^{\mu a} e_\mu \\
Q_a e_\mu = F_\beta^{\mu a} e_b - F_\beta^{\mu b} e_{a+b+m} - 2F_\alpha^{\mu a} e_\alpha
\]

where the coefficients are defined now in (4.18) and (8.4). We first outline the quite elementary proof of the theorem, and then follow that with a proof of the details. The first detail is:

**(I).** At each point of \( U \) these operators are symmetric, orthogonal and satisfy

\[
(9.5) \quad Q_i Q_j + Q_j Q_i = 2\delta_{ij} I, \quad \text{for } i, j = 0, 1, \ldots, m
\]

Given that, one next proves the second detail:
(II). There exist a (constant) Clifford system $P_0, \ldots, P_m$ on $\mathbb{R}^{n+1}$ and a smooth map

$$(9.6) \quad B : U \to SO(m+1)$$

such that at every point of $U$,

$$(9.7) \quad Q_j = \sum_{i=0}^{m} B^i_j P_i, \quad \text{for } j = 0, 1, \ldots, m$$

It will then follow that $\mathbf{x}$ maps $U$ onto an open subset of the focal submanifold $M_+$ defined in (7.4) by this Clifford system, and that the Darboux frame field (7.14) coming from the FKM construction applied to $P_0, \ldots, P_m$ coincides with our frame field (9.1). Therefore, our $\mathbf{x} : U \to S^n$ coincides with the FKM construction applied to this Clifford system.

We turn now to the proof of detail (I). The verification that each $Q_i$ is symmetric can be done almost by inspection. It is equally clear that $Q_0$ is orthogonal, since it sends the orthonormal basis (9.1) to an orthonormal basis. The operator $Q_a$ sends the orthonormal basis (9.1) to the set of vectors given on the right hand side of (9.4). Among these vectors, $Q_a \mathbf{x}, Q_a e_0$ is an orthonormal pair orthogonal to the remaining vectors because $L^a_{bc}$ are skew-symmetric in $a, b, c$ and $F^\alpha_{a+m} b$ and $F^\alpha_{a+m} b$ are skew-symmetric in $a$ and $b$.

In the following verification that

$$\{Q_a e_b, Q_a e_{b+m}, Q_a e_a, Q_a e_\mu : b, a, \mu\}$$

is orthonormal, we do not use the Einstein summation convention as $a$ will always be a repeated index which is not summed. We proceed through all the cases.

$$Q_a e_b \cdot Q_a e_d = \delta_{ab} \delta_{ad} + \sum_c L^b_{ac} L^d_{ac}$$

$$+ \sum_\alpha F^\alpha_{a+m} b F^\alpha_{a+m} d + \sum_\mu F^\mu_{a+m} b F^\mu_{a+m} d = \delta_{bd}$$

by (8.55) with $c$ changed to $b$ and $b$ changed to $a$.

$$Q_a e_b \cdot Q_a e_{d+m} = \sum_\alpha F^\alpha_{a+m} b F^\alpha_{d+m} a - \sum_\mu F^\mu_{a+m} b F^\mu_{d+m} a = 0$$

by the first equation in (5.6).

$$Q_a e_b \cdot Q_a e_a = \sum_c L^b_{ac} F^\alpha_{c+m} a - 2 \sum_\mu F^\mu_{a+m} b F^\mu_{aa} a = 0$$
by (8.37) with $d$ changed to $a$.

$$Q_a e_b \cdot Q_a e_{\mu} = - \sum_c L^b_{ac} F^\mu_{c+m} - 2 \sum_\alpha F^\alpha_{a+m} b F^\mu_{\alpha a} = 0$$

by (8.38) with $d$ changed to $a$.

$$Q_a e_{b+m} \cdot Q_a e_{d+m} = \delta_{ab} \delta_{ad} + \sum_c L^c_{ab} L^c_{ad}$$

$$+ \sum_\alpha F^\alpha_{b+m} a F^\alpha_{a+m} + \sum_\mu F^\mu_{b+m} a F^\mu_{d+m} = \delta_{bd}$$

by (8.55) with $c$ changed to $b$ and $b$ changed to $a$.

$$Q_a e_{b+m} \cdot Q_a e_{a} = \sum_c L^c_{ab} F^\alpha_{a+m c} + 2 \sum_\mu F^\mu_{b+m} a F^\mu_{a a} = 0$$

by (8.37) with $d$ changed to $a$.

$$Q_a e_{b+m} \cdot Q_a e_{\mu} = \sum_c L^c_{ab} F^\mu_{a+m c} - 2 \sum_\alpha F^\alpha_{b+m} a F^\alpha_{a a} = 0$$

by (8.38) with $d$ changed to $a$.

$$Q_a e_{a} \cdot Q_a e_{\beta} = 2 \sum_b F^\alpha_{b+m} F^\beta_{b+m} + 4 \sum_\mu F^\mu_{a a} F^\mu_{\beta a} = \delta_{\alpha \beta}$$

by the second equation in (5.6).

$$Q_a e_{\alpha} \cdot Q_a e_{\mu} = \sum_b F^\alpha_{a+m b} F^\mu_{a+m b} - \sum_b F^\alpha_{b+m a} F^\mu_{b+m a} = 0$$

by (8.2) and (8.3).

$$Q_a e_{\mu} \cdot Q_a e_{\nu} = 2 \sum_b F^\mu_{b+m a} F^\nu_{b+m a} + 4 \sum_\alpha F^\mu_{a a} F^\nu_{a a} = \delta_{\mu \nu}$$

by the fourth equation in (5.6). That completes the verification that each $Q_i$ is an orthogonal transformation.

We proceed now to verify (9.5). For this we return to using the Einstein summation convention. Clearly $Q_0^2 = I$. To verify that $Q_0 Q_a + Q_a Q_0 = 0$, for all $a$, we set $S = Q_0 Q_a + Q_a Q_0$ and evaluate it on the basis vectors.

$$S x = Q_0 e_a + Q_a e_0 = -e_{a+m} + e_{a+m} = 0$$

$$S e_0 = Q_0 e_{a+m} + Q_a x = -e_a + e_a = 0$$

$$S e_b = Q_0 (\delta_{ab} x + L^b_{ac} e_{c+m} + F^\alpha_{a+m b} e_{\alpha} + F^\mu_{a+m b} e_{\mu}) + Q_a (-e_b + m)$$

$$= \delta_{ab} e_0 - L^b_{ac} e_{c} - F^\alpha_{a+m b} e_{\alpha} + F^\mu_{a+m b} e_{\mu}$$

$$- \delta_{ab} e_0 - L^c_{ab} e_{c} - F^\alpha_{b+m a} e_{\alpha} + F^\mu_{b+m a} e_{\mu} = 0$$
Next we verify that $T = Q_a Q_d + Q_d Q_a$ and we evaluate it on the basis vectors.

$$T e_0 = Q_a e_{d+m} + Q_d e_a = \delta_{ad} e_0 + L_{ae} e_{c+m} + F_{b+m a} e_a - F_{a+m d} e_{c+m} + F_{a+m d} e_{c+m} = 2\delta_{ad} e_0$$

$$T e_b = Q_a (\delta_{bd} e_c + L_{dc} e_{c+m} + F_{d+m a} e_a + F_{d+m b} e_{c+m}) + Q_d (\delta_{ad} e_b + L_{be} e_{c+m} + F_{a+m b} e_a + F_{a+m b} e_{c+m})$$

where the coefficient of $e_e$ comes from (8.55), the coefficient of $e_{e+m}$ is zero by the first equation of (5.6), the coefficient of $e_{\alpha}$ is zero by (8.37) (with the roles of $b$ and $d$ reversed) and the coefficient of $e_{\mu}$ is zero.
by (8.38) (with the roles of $b$ and $d$ reversed).

\[ T_{e_{b+m}} = (L_{db}^a + L_{ab}^d)x \]
\[ + (F_{b+m}^a F_{b+m}^a + F_{b+m}^b F_{b+m}^b - F_{b+m}^c F_{b+m}^c - F_{b+m}^d F_{b+m}^d) e_c \]
\[ + (\delta_{db} \delta_{ae} + \delta_{ab} \delta_{de} + L_{db}^c L_{ae}^c + L_{ab}^c L_{de}^c) \]
\[ + F_{b+m}^a F_{b+m}^a + F_{b+m}^b F_{b+m}^b + F_{b+m}^c F_{b+m}^c + F_{b+m}^d F_{b+m}^d) e_c \]
\[ + (L_{db}^c F_{b+m}^a + L_{ab}^c F_{b+m}^a + 2F_{b+m}^a F_{b+m}^a + 2F_{b+m}^a F_{b+m}^a) e_\alpha \]
\[ + (L_{db}^c F_{b+m}^a + L_{ab}^c F_{b+m}^a - 2F_{b+m}^a F_{b+m}^a - 2F_{b+m}^a F_{b+m}^a) e_\mu \]
\[ = 2\delta_{ad} \delta_{be} e_{c+m} = 2\delta_{ad} e_{b+m} \]

where the coefficient of $e_{c+m}$ comes from (8.55), the coefficient of $e_c$ is zero by the first equation of (5.6), the coefficient of $e_\alpha$ is zero by (8.37) (with the roles of $b$ and $d$ reversed) and the coefficient of $e_\mu$ is zero by (8.38) (with the roles of $b$ and $d$ reversed).

\[ T_{e_\alpha} = (F_{d+m}^a + F_{a+m}^d)x + (F_{a+m}^d + F_{d+m}^a) e_0 \]
\[ + (F_{b+m}^a L_{db}^a + F_{b+m}^b L_{ab}^c - 2F_{ad}^a F_{a+m}^c - 2F_{ad}^a F_{d+m}^c) e_c \]
\[ + (F_{b+m}^a L_{db}^a + F_{b+m}^b L_{ab}^c - 2F_{ad}^a F_{a+m}^c - 2F_{ad}^a F_{d+m}^c) e_c \]
\[ + 2(F_{b+m}^a F_{b+m}^a + F_{b+m}^b F_{b+m}^b + 2F_{ad}^a F_{b+m}^a + 2F_{ad}^a F_{b+m}^a) e_\beta \]
\[ + (F_{b+m}^a F_{b+m}^a + F_{b+m}^b F_{b+m}^b - F_{b+m}^a F_{b+m}^a - F_{b+m}^a F_{b+m}^a) e_\mu \]
\[ = 2\delta_{ad} \delta_{be} e_{c+m} = 2\delta_{ad} \delta_{be} e_{b+m} \]

where the coefficients of $x$ and $e_0$ are clearly zero, the coefficients of $e_c$ and of $e_{c+m}$ are zero by (8.37) (in which the roles of $a, b, c, d$ are here played by $a, d, b, c$), the coefficient of $e_\beta$ comes from the second equation of (5.6) and the coefficient of $e_\mu$ is clearly zero.

\[ T_{e_\mu} = (F_{d+m}^a + F_{a+m}^d)x - (F_{a+m}^d + F_{d+m}^a) e_0 \]
\[ - (2(F_{ad}^a F_{a+m}^b + F_{ad}^a F_{d+m}^b) + F_{c+m}^a L_{ac}^b + F_{c+m}^a L_{ac}^b) \]
\[ + (F_{d+m}^a L_{ac}^b + 2F_{ad}^a F_{b+m}^a + F_{ad}^a L_{ac}^b + 2F_{ad}^a F_{b+m}^a) e_{b+m} \]
\[ + (F_{d+m}^a F_{a+m}^b - F_{b+m}^a F_{b+m}^a + F_{a+m}^b F_{d+m}^a - F_{b+m}^a F_{b+m}^a) e_\alpha \]
\[ + 2(F_{d+m}^a F_{a+m}^b + 2F_{ad}^a F_{d+m}^b + F_{ad}^a F_{d+m}^b + 2F_{ad}^a F_{d+m}^b) e_\mu \]
\[ = 2\delta_{ad} \delta_{be} e_{c+m} = 2\delta_{ad} \delta_{be} e_{b+m} \]

where the coefficients of $x$ and $e_0$ are clearly zero, the coefficients of $e_b$ and $e_{b+m}$ are zero by (8.38) (in which the roles of $b$ and $d$ are reversed), the coefficient of $e_\alpha$ is zero by (8.2) and (8.3), and the coefficient of $e_\mu$ comes from the fifth equation of (5.6). This completes the proof of detail (I).
In order to prove (II), we must find a Clifford system \( P_0, \ldots, P_m \) which is related to \( Q_0, \ldots, Q_m \) by (9.7). We do this by finding the map \( B : U \to SO(m + 1) \) of (9.6). Let
\[
(9.8) \quad \nu^a = \omega^a + \omega^{a+m}
\]
Use (5.1) together with (8.2)–(8.4) to find
\[
(9.9) \quad d\nu^a = -\nu^a_c \wedge \nu^b
\]
where
\[
(9.10) \quad \nu^a_b = \theta^a_b + L^a_{cb} \omega^c + F^a_{a+m b} \omega^a + F^\mu_{a+m b} \omega^\mu = -\nu^a_b
\]
Set
\[
(9.11) \quad \nu^0_b = -\nu^b_a = -\nu^b = - (\omega^b + \omega^{b+m})
\]
We shall verify below that
\[
(9.12) \quad dQ_j = \sum_{k=0}^m Q_k \nu^k_j, \text{ for } j = 0, \ldots, m
\]
Differentiating this, we find that
\[
(9.13) \quad d\nu^i = -\sum_{k=0}^m \nu^i_k \wedge \nu^b_j, \text{ for } i, j = 0, \ldots, m
\]
In fact, (9.9) is the case \( i = a, j = 0 \) and also implies the case \( i = 0, j = a \). To verify the remaining cases in (9.13), we take the exterior derivative of (9.12) when \( j = a \), and then use (9.12) and (9.9) to find
\[
0 = ddQ_a = dQ_0 \wedge \nu^0_a + Q_0 d\nu^0_a + dQ_b \wedge \nu^b_a + Q_b d\nu^b_a
\]
\[
(9.14) \quad = Q_b \nu^b \wedge \nu^0_a + Q_0 \nu^0_b \wedge \nu^b + (Q_0 \nu^0_b + Q_b \nu^0_a) \wedge \nu^b_a + Q_b d\nu^b_a
\]
\[
= Q_b (d\nu^b_a + \nu^b_c \wedge \nu^c_a + \nu^b \wedge \nu^0_a) + Q_0 (\nu^0_b \wedge \nu^b + \nu^0_b \wedge \nu^b_a)
\]
which implies (9.13) because the coefficient of \( Q_0 \) is zero and the \( Q_b \) are linearly independent at each point of \( U \), as can be seen from the fact that \( Q_b x = e_b \) are linearly independent at each point. Define the \( o(m+1) \)-valued 1-form \( \nu \) to be
\[
(9.15) \quad \nu = \begin{pmatrix} 0 & \nu^0_b \\ \nu^0_a & \nu^a_b \end{pmatrix}
\]
Then (9.9) and (9.13) imply that \( d\nu = -\nu \wedge \nu \). Therefore, on a simply connected subset of \( U \), which we continue to call \( U \), there exists a smooth map
\[
(9.16) \quad A : U \to SO(m + 1)
\]
such that $A^{-1} dA = \nu$. Denote the entries of $A$ by the functions $A^i_j$, $i, j = 0, \ldots, m$, so that the entries of $dA = A \nu$ are given by

$$
(9.17) \quad dA^i_j = \sum_{k=0}^{m} A^i_k \nu^k_j
$$

Let

$$
(9.18) \quad P_i = \sum_{j=0}^{m} A^i_j Q_j, \text{ for } i = 0, \ldots, m
$$

which, at each point of $U$, is a set of symmetric, orthogonal transformations of $\mathbb{R}^{n+1}$ satisfying the conditions $P_i P_j + P_j P_i = 2 \delta_{ij} I$, since $Q_i Q_j + Q_j Q_i = 2 \delta_{ij} I$ and $A \in SO(m+1)$. Using (9.12) and (9.17), we have

$$
(9.19) \quad dP_i = \sum_{j=0}^{m} \left( (dA^i_j) Q_j + A^i_j dQ_j \right) = \sum_{j,k=0}^{m} \left( A^i_k Q_j \nu^k_j + A^i_j Q_k \nu^k_j \right) = 0
$$

since $\nu^i_j + \nu^j_i = 0$. Therefore, each $P_i$ is constant on $U$ and $P_0, \ldots, P_m$ define a Clifford system on $\mathbb{R}^{n+1}$ and (9.7) holds with $B = A^{-1}$.

All that remains of the proof of detail (II) is to verify (9.12), for which we need the Maurer-Cartan equations (4.20) for our Darboux frame field. We first verify (9.12) for $j = 0$, then for $j = a$, in both cases by evaluating each side on the basis vectors. Differentiating equations in (9.3) and using (4.20), we get

$$
(dQ_0) x = d(Q_0 x) - Q_0 \, dx
$$

$$
= de_0 - Q_0 (\omega^a e_a + \omega^e e_e + \omega^\mu e_\mu)
= \omega^a e_a - \omega^a e_a + \omega^\mu e_\mu + \omega^a e_a - \omega^\mu e_\mu
= \nu^a e_a = \nu^a Q_a x
$$

$$
(dQ_0) e_0 = d(Q_0 e_0) - Q_0 \, de_0
$$

$$
= dx - Q_0 (\omega^a e_a - \omega^a e_a + \omega^\mu e_\mu)
= \nu^a e_{a+m} = \nu^a Q_a e_0
$$

$$
(dQ_0) e_a = d(Q_0 e_a) - Q_0 \, de_a = -de_{a+m} - Q_0 \, de_a
$$

$$
= \nu^a x + (\omega^b x - \omega^b x) e_b + (\omega^b x - \omega^b x) e_{b+m}
+ (\omega^a x - \omega^a x) e_a - (\omega^a x - \omega^a x) e_\mu
= \nu^b (\delta_{ab} x + L^c_{ab} e_{c+m} + F^a_{b+m a} e_a + F^\mu_{b+m a} e_\mu)
= \nu^b Q_b e_a
$$
\((dQ_0)e_{a+m} = d(Q_0e_{a+m}) - Q_0 de_{a+m} = -de_a - Q_0 de_{a+m}
\]
\[= (\omega^a + \omega^{a+m})e_a + (\omega^{b+m}_a - \omega^b_a)e_b + (\omega^{b+m}_a - \omega^{b+m}_a)e_{b+m}
\]
\[+ (\omega^{a+m}_a - \omega^a_a)e_a - (\omega^a_a + \omega^b_a)e_e
\]
\[= \nu^b(\delta_{ab}e_0 - L^c_{ab}e_c + F^\alpha_{a+m}b e_\alpha - F^\mu_{a+m}b e_\mu) = \nu^b Q_b e_{a+m}
\]
\[(dQ_0)e_a = d(Q_0e_a) - Q_0 de_a = -de_a - Q_0 de_a
\]
\[= (\omega^{a+m}_a - \omega^a_a)e_a + (\omega^a_a - \omega^{a+m}_a)e_{a+m} - 2\omega^\mu e_\mu
\]
\[= \nu^b(F^\alpha_{b+m}a e_a - F^\alpha_{b+m}a e_{a+m} - 2F^\mu_{a+b}e_\mu) = \nu^b Q_b e_a
\]
\[(dQ_0)e_\mu = d(Q_0e_\mu) - Q_0 de_\mu = de_\mu - Q_0 de_\mu
\]
\[= (\theta^a_\mu + \theta^{a+m}_\mu)(e_a + e_{a+m}) + 2\theta^\mu e_a
\]
\[= \nu^b(F^\mu_{b+m}a (e_a + e_{a+m}) - 2F^\mu_{a+b}e_\mu) = \nu^b Q_b e_\mu
\]

That completes the verification of (9.12) for the case \(j = 0\).

We now verify the equations in (9.12) for the cases \(j = a\) by applying each side to the basis vectors. Using (9.4) and (4.20), we have

\[(dQ_a)x = d(Q_ax) - Q_adx = de_a - Q_a(\omega^{b+m}_a e_{b+m} + \omega^a e_a + \omega^\mu e_\mu)
\]
\[= -\nu^a e_0 + (\theta^\mu_a - L^b_a e^c e_a - F^\alpha_{a+m}b e_\alpha - F^\mu_{a+m}b e_\mu)e_b
\]
\[+ (\theta^{b+m}_a - F^\alpha_{b+m}a e_a - F^\mu_{b+m}a e_\mu)e_{b+m}
\]
\[+ (\theta^a_a - F^\alpha_{b+m}a e_a + 2F^\mu_{a+b}e_\mu)e_a + (\theta^a_a + F^\mu_{a+b}a e^b + 2F^\mu_{a+b} e_\mu)e_\mu
\]
\[= -\nu^a e_0 + \nu^b e_b = (\nu^a Q_b + \nu^b Q_b)x
\]

\[(dQ_a)e_0 = d(Q_ax) - Q_ad0 = de_{a+m} - Q_a(\omega^b e_b - \omega^a e_a + \omega^\mu e_\mu)
\]
\[= (\omega^{a+m}_a - \omega^a_a)x + (\theta^{b+m}_a + F^\alpha_{a+m}b e_\alpha + F^\mu_{a+m}b e_\mu)e_b
\]
\[+ (\theta^a_a - F^\alpha_{a+m}b e_a + 2F^\mu_{a+b}e_a)e_a
\]
\[+ (\theta^a_a - F^\mu_{a+m}b e^b - 2F^\mu_{a+b} e_\mu)e_\mu
\]
\[= -\nu^a x + (\nu^b e_b) = (\nu^a Q_0 + \nu^b Q_0)e_0
\]

where the coefficients of \(e_b, e_a\) and \(e_\mu\) are zero by (4.18), and (9.10) is used in the coefficient of \(e_{b+m}\).

In order to verify (9.12) when both sides are applied to \(e_b\), we must verify that

\[d(Q_a e_b) - Q_a de_b = (dQ_a)e_b - \nu^a_0 Q_0 e_b + \nu^b_0 Q_c e_b
\]
\[= \nu^b_0 x + (\delta_{bd}\nu^a + L^c_{ab}\nu^e_a)e_{d+m} + F^\alpha_{c+m}b \nu^e_a e_\alpha + F^\mu_{c+m}b \nu^e_\mu
\]

\[(9.20)\]
Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get

\[
d(Q_a e_b) - Q_a d e_b = \\
(L^c_{ab} \omega^{c+m} - F^\alpha_{a+m b} \omega^\alpha - F^\mu_{a+m b} \omega^\mu - \theta^a_b) x \\
+ (L^c_{a+m b} \omega^\alpha - F^\mu_{a+m b} \omega^\mu - \theta^a_b) e_0 \\
+ (-L^c_{ab} \theta^d_{c+m} + F^\alpha_{a+m b} \theta^d_{\alpha} + F^\mu_{a+m b} \theta^d_{\mu}) \\
- L^d_{ac} \theta^b_{c+m} - F^\alpha_{a+m d} \theta^b_{\alpha} - F^\mu_{a+m d} \theta^b_{\mu}) e_0 \\
(9.21)
\]

\[
+ (\delta_{ab} \omega^{c+m} - d L^c_{ab} - L^d_{a+m} \theta^c_{d+m} + F^\alpha_{a+m b} \theta^c_{a} + F^\mu_{a+m b} \theta^c_{\mu} \\
+ \delta_{ac} \omega^b + L^c_{ad} \theta^d_{b} - F^\alpha_{c+m a} \theta^d_{\alpha} + F^\mu_{c+m a} \theta^d_{\mu}) e_c + m \\
+ (\delta_{ab} \omega^a - L^c_{ab} \theta^a_{c+m} + d F^\alpha_{a+m b} + F^\beta_{a+m b} \theta^a_{\beta} \\
+ F^\mu_{a+m b} \theta^a_{\mu} - F^\alpha_{a+m b} \theta^a_{\alpha} - F^\mu_{a+m b} \theta^a_{\mu} + 2 F^\mu_{a+m b} \theta^a_{\mu}) e_a \\
+ (\delta_{ab} \omega^\mu - L^c_{ab} \theta^\mu_{c+m} + F^\alpha_{a+m b} \theta^\mu_{\alpha} + d F^\mu_{a+m b} \\
+ F^\mu_{a+m b} \theta^\alpha_{b} + F^\mu_{c+m a} \theta^\mu_{c+m} + 2 F^\mu_{a+m b} \theta^\mu_{b}) e_\mu
\]

The coefficient of \( x \) is \( \nu^a_a \) by (9.10). The coefficient of \( e_0 \) is zero. Substituting (4.18) into the coefficient of \( e_d \), we get

\[
(-F^\alpha_{a+m b} F^\alpha_{c+m d} - F^\alpha_{a+m d} F^\alpha_{c+m b} + F^\mu_{a+m b} F^\mu_{c+m d} + F^\mu_{a+m d} F^\mu_{c+m b}) \omega^{c+m} \\
- (L^c_{ac} F^\alpha_{c+m b} + L^c_{ac} F^\alpha_{c+m d} - 2 F^\alpha_{a+m b} F^\alpha_{a+m d} - 2 F^\alpha_{a+m d} F^\alpha_{a+m b}) \omega^\alpha \\
+ (L^c_{ac} F^\mu_{c+m b} + L^c_{ac} F^\mu_{c+m d} + 2 F^\mu_{a+m b} F^\mu_{a+m d} + 2 F^\mu_{a+m d} F^\mu_{a+m b}) \omega^\mu
\]

which is zero since the coefficient of \( \omega^{c+m} \) is zero by the first equation in (5.6), the coefficient of \( \omega^\alpha \) is zero by (8.37) and the coefficient of \( \omega^\mu \) is zero by (8.38). Thus, the coefficient of \( e_d \) is zero, in agreement with the right hand side of (9.20).

By (4.18), (8.50) and (9.8) with (9.10), the coefficient of \( e_{c+m} \) becomes

\[
-L^c_{db} \nu^d_a + \nu^a \delta^c_b \\
- (L^c_{ab} - L^c_{ab} L^c_{ed} + F^\alpha_{a+m b} F^\alpha_{c+m d} + F^\mu_{a+m b} F^\mu_{c+m d} - \delta_{ac} \delta_{bd} + \delta_{bc} \delta_{ad}) \omega^d \\
+ (-L^c_{ab} F^\alpha_{c+m a} F^\alpha_{c+m b} - L^c_{ac} L^c_{da} + L^c_{ab} L^c_{ed} - F^\alpha_{c+m a} F^\alpha_{c+m b} + F^\mu_{c+m a} F^\mu_{c+m b} - \delta_{ac} \delta_{bd} - \delta_{bc} \delta_{ad}) \omega^d \\
+ (F^\mu_{c+m a} F^\mu_{c+m b} + \delta_{bc} \delta_{ad} - \delta_{bc} \delta_{ad}) \omega^d \\
+ (L^c_{ab} - L^c_{ab} F^\alpha_{c+m a} - F^\mu_{a+m b} F^\alpha_{c+m b} - 2 F^\mu_{a+m b} F^\mu_{c+m a} - 2 F^\mu_{c+m a} F^\mu_{a+m b}) \omega^\alpha \\
+ (-L^c_{ab} + L^c_{ab} F^\alpha_{c+m a} + F^\mu_{a+m b} F^\mu_{c+m a} + 2 F^\mu_{a+m b} F^\mu_{c+m a} F^\mu_{ab}) \omega^\mu
\]

We now verify that zero is the coefficient of each of \( \omega^d, \omega^{d+m}, \omega^\alpha, \omega^\mu \).
The coefficient of $\omega^d$ can be seen to be zero by taking (8.51) (with indices in the order $c, a, b, d$) and subtracting half of (8.55) (with indices as is).

The coefficient of $\omega^{d+m}$ can be seen to be zero by using (8.60), then adding (8.51) (with indices in the order $c, a, b, d$), then adding half of (8.55) (with indices as is), and then using (8.55) again (with the roles of $d$ and $c$ reversed).

The coefficient of $\omega^e$ can be seen to be zero from (8.53) and (8.37).

The coefficient of $\omega^\mu$ is zero by (8.54).

Hence, we have shown that the coefficient of $\epsilon_{c+m}$ in $(dQ_a)e_b$ is as given in (9.20).

Using (5.2), (8.4) and (9.10), we can rewrite the coefficient of $\epsilon_a$ in $(dQ_a)e_b$ in (9.21) as

\[
F^\alpha_{c+m}b^\alpha_c + (F^\alpha_{a+m}b + L^b_{ad}F^\alpha_{d+m}c - F^\mu_{a+m}bF^\mu_{ac} - F^\alpha_{d+m}b L^d_{ac})\omega^c \\
+ (F^\alpha_{a+m}b + L^b_{ad}F^\alpha_{d+m}c - F^\mu_{a+m}bF^\mu_{ac} - F^\alpha_{d+m}b L^d_{ac})\omega^c \\
+ (F^\alpha_{a+m}b + \delta_{ab}\delta_{\alpha\beta} - F^\alpha_{c+m}bF^\beta_{c+m}b - 4F^\alpha_{a+m}bF^\mu_{c+m}a)\omega^\beta \\
+ (F^\alpha_{a+m}b - 2L^b_{ac}F^\alpha_{ac} + F^\alpha_{c+m}bF^\mu_{c+m}b - F^\alpha_{c+m}bF^\mu_{c+m}a)\omega^\mu
\]

The coefficient of $\omega^c$ is seen to be zero by using the first equation in (5.7) and then using (8.37). The coefficient of $\omega^{c+m}$ is zero by the second equation in (5.7). The coefficient of $\omega^\beta$ is seen to be zero by using the third equation in (5.7) and then using the second equation in (5.6). The coefficient of $\omega^\mu$ is seen to be zero by using (5.11) and then using (8.39) (with the roles of $a$ and $b$ interchanged).

Using (5.2) and (9.10), we can rewrite the coefficient of $\epsilon_\mu$ in $(dQ_a)e_b$ in (9.21) as

\[
F^\mu_{c+m}b^\nu_c + (F^\mu_{a+m}b + L^b_{ad}F^\mu_{d+m}c - F^\nu_{a+m}bF^\nu_{ac} - F^\mu_{d+m}b L^d_{ac})\omega^c \\
+ (F^\mu_{a+m}b + L^b_{ad}F^\mu_{d+m}c - F^\nu_{a+m}bF^\nu_{ac} - F^\mu_{d+m}b L^d_{ac})\omega^c \\
+ (F^\mu_{a+m}b + \delta_{ab}\delta_{\mu\nu} - F^\mu_{c+m}bF^\nu_{c+m}b - 4F^\mu_{a+m}bF^\nu_{c+m}a)\omega^\nu \\
+ (F^\mu_{a+m}b - 2L^b_{ac}F^\mu_{ac} + F^\mu_{c+m}bF^\nu_{c+m}b - F^\mu_{c+m}bF^\nu_{c+m}a)\omega^\nu
\]

The coefficient of $\omega^c$ is seen to be zero by using the first equation in (5.8) and then using (8.38). The coefficient of $\omega^{c+m}$ is zero by the second equation in (5.8). The coefficient of $\omega^\mu$ is zero by (8.39). The coefficient of $\omega^\alpha$ is seen to be zero by using the third equation in (5.8) and then using the fourth equation in (5.6). This completes the verification of (9.20).
The next case is to verify (9.12) when both sides are applied to \( e_{b+m} \).
We must verify that

\[
(9.22) \quad d(Q_a e_{b+m}) - Q_a d e_{b+m} = (dQ_a) e_{b+m} = \nu^b_a Q_0 e_{b+m} + \nu^c_a Q_c e_{b+m}
\]

\[
= \nu^b_a e_b + \nu^b_a e_0 + L_{db} \nu^d_a e_c + F^\alpha_{b+m} \nu^\alpha_a e_\alpha - F^\mu_{b+m} \nu^\mu_a e_\mu
\]

by (9.4). Using (9.4) to compute \( Q_a e_{b+m} \) and (4.20) to compute \( d e_{b+m} \), the left hand side becomes

\[
\begin{align*}
& (-F^\alpha_{b+m} \omega^\alpha + F^\mu_{b+m} \omega^\mu - \theta^\mu_{b+m}) x \\
& \quad \quad + (-L^c_{ab} \omega^c + F^\alpha_{b+m} \omega^\alpha + F^\mu_{b+m} \omega^\mu - \theta^\alpha_{b+m}) e_0 \\
& \quad \quad + (\delta_{ab} \omega^c + dF^c_{ab} + F^\alpha_{b+m} \theta^\alpha_{ab} - F^\mu_{b+m} \theta^\mu_{ab}) \\
& \quad \quad \quad + \delta_{ac} \omega^b + L^d_{ad} \theta^d_{b+m} - F^\alpha_{a+m} \theta^\alpha_{b+m} - F^\mu_{a+m} \theta^\mu_{b+m}) e_c \\
& \quad \quad + (L^c_{ab} \theta^c_{b+m} + F^\alpha_{b+m} \theta^\alpha_{ab} - F^\mu_{b+m} \theta^\mu_{ab}) e_{d+m} \\
& \quad \quad \quad + (-\delta_{ab} \omega^c + L^c_{ab} \theta^c_{ab} + F^\alpha_{b+m} \theta^\alpha_{ab} - F^\mu_{b+m} \theta^\mu_{ab}) e_{d+m} \\
& \quad \quad \quad \quad - F^\alpha_{a+m} \theta^\alpha_{b+m} - F^\mu_{a+m} \theta^\mu_{b+m} + 2F^\mu_{a+d} \theta^\mu_{b+m} e_\alpha \\
& \quad \quad \quad \quad + (\delta_{ab} \omega^\mu + L^c_{ab} \theta^\mu_{ab} + F^\alpha_{b+m} \theta^\alpha_{ab} - dF^\mu_{b+m} a - F^\mu_{b+m} \theta^\mu_{ab}) \\
& \quad \quad \quad \quad \quad - F^\mu_{a+m} \theta^\mu_{b+m} + F^\mu_{c+m} \theta^\mu_{b+m} + 2F^\mu_{a+m} \theta^\mu_{b+m} e_\mu
\end{align*}
\]

\[
(9.23)
\]

We want to verify that this is equal to the right side of (9.22), where \( \nu^c_a \) is given by (9.10). We do this by comparing the coefficients of the basis vectors \( x, e_0, e_c, e_{c+m}, e_\alpha, e_\mu \).

The coefficient of \( x \) is 0 by (4.18).

The coefficient of \( e_0 \) is

\[
\theta^b_a + L^b_{ca} \omega^{c+m} + F^\alpha_{b+m} \omega^\alpha + F^\mu_{b+m} \omega^\mu - L^b_{ca} (\omega^c + \omega^{c+m}) + (\theta^b_{a+m} - \theta^b_a)
\]

by (8.4), (9.8) and (9.10).

The coefficient of \( e_c \) in \( (dQ_a) e_{b+m} \) in (9.23) is, using (4.18) and (8.50) and the skew-symmetry of \( L^a_{bcd} \) in all four indices,

\[
L^d_{ab} (\theta^d_a + L^e_{ea} \omega^{e+m} + F^\alpha_{e+m} \omega^\alpha + F^\mu_{e+m} \omega^\mu) + (\omega^a + \omega^{a+m} ) \delta_{bc}
\]

\[
+ (L^b_{cd} + \delta_{ab} \delta_{cd} - \delta_{da} \delta_{bc} + L^e_{ac} L^d_{bd} - F^\alpha_{a+m} F^\alpha_{b+m} a - F^\mu_{a+m} F^\mu_{b+m} a) \omega^d
\]

\[
+ (L^c_{ab} d + \delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc} + L^e_{eb} L^d_{ad} + L^e_{ac} L^d_{bd} + F^\alpha_{a+m} F^\alpha_{b+m} a - F^\mu_{a+m} F^\mu_{b+m} a) \omega^d
\]

\[
= L^d_{db} \nu^d_a + \nu^d_a \delta_{bc}
\]
by (9.8) and (9.10) and the following. Using (8.45) for $L^a_{bcd}$, the coefficient of $\omega^d$ becomes

$$
\delta_{ab}\delta_{cd} - \frac{1}{2}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{1}{2}(L^c_{ae}L^e_{bd} - L^e_{ae}L^e_{bd}) \\
+ F^\alpha_{c+m a}F^\mu_{b+m a} - F^\mu_{c+m a}F^\alpha_{b+m a}
$$

which is zero by (8.49) combined with the first equation of (5.6). In the coefficient of $\omega^{d+m}$, substitute (8.46) for $L^a_{bcd+m} = L^e_{ab d+m}$, and gather together terms using skew symmetries, to get

$$
\frac{3}{2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + L^a_{be}L^d_{cc} + \frac{1}{2}L^a_{de}L^e_{bc} + \frac{1}{2}L^a_{ce}L^d_{bc} \\
- F^\alpha_{a+m d}F^\alpha_{c+m b} - F^\alpha_{a+m b}F^\alpha_{c+m d} + F^\mu_{d+m b}F^\mu_{a+m c} + F^\mu_{a+m b}F^\mu_{d+m c}
$$

which is zero by using the first equation in (5.6) and then (8.49). The coefficient of $\omega^c$ is zero by (8.47). The coefficient of $\omega^\alpha$ is zero by (8.48).

The coefficient of $e_{d+m}$ in $(dQ_\alpha)e_{b+m}$ in (9.23) is, using (4.18)

$$
(-F^\alpha_{b+m a}F^\alpha_{d+m c} + F^\alpha_{b+m a}F^\mu_{d+m c} + F^\alpha_{d+m a}F^\alpha_{b+m c} + F^\mu_{d+m a}F^\alpha_{b+m c})\omega^c \\
+ (L^\alpha_{ba}F^\alpha_{c+m a} - 2F^\alpha_{a+m b}F^\mu_{a+m c} + L^\alpha_{da}F^\alpha_{c+m b} - 2F^\mu_{a+m d}F^\alpha_{a+m b})\omega^\alpha \\
+ (-L^\alpha_{ba}F^\mu_{c+m a} + 2F^\alpha_{b+m a}F^\alpha_{d+m c} - L^\alpha_{da}F^\mu_{c+m b} + 2F^\alpha_{d+m a}F^\alpha_{a+m b})\omega^\mu
$$

$$
= 0
$$

because the coefficient of $\omega^c$ is 0 by the first equation in (5.6), the coefficient of $\omega^\alpha$ is 0 by (8.37) and (8.1), and the coefficient of $\omega^\mu$ is 0 by (8.38) and (8.1).

The coefficient of $e_\alpha$ in $(dQ_\alpha)e_{b+m}$ in (9.23) is, using (4.18) and (5.2)

$$
F^\alpha_{b+m c}(\theta^\alpha_a + L^\alpha_{da}\omega^{d+m} + F^\beta_{c+m a}\omega^\beta + F^\mu_{c+m a}\omega^\mu) \\
+ (F^\alpha_{b+m a}F^\mu_{c+m a} + 2F^\alpha_{a+m c}F^\mu_{b+m c})\omega^c \\
+ (F^\alpha_{b+m a}F^\alpha_{d+m c} + F^\alpha_{c+m a}F^\mu_{d+m c} + F^\mu_{b+m a}F^\alpha_{d+m c})\omega^{d+m} \\
+ (-\delta_{ab}\delta_{\alpha\beta} + F^\alpha_{b+m a}\delta_{\alpha\beta} + F^\alpha_{a+m c}\delta_{\beta\alpha} - F^\mu_{b+m c}\delta_{\beta\alpha}) + 4F^\alpha_{a+m c}F^\mu_{b+m c})\omega^{d+m} \\
+ (-2L^\alpha_{ba}F^\mu_{c+m a} - F^\alpha_{b+m a}\delta_{\alpha\mu} - F^\alpha_{a+m c}F^\mu_{b+m c} - F^\mu_{b+m c}F^\alpha_{c+m a})\omega^\mu
$$

$$
= F^\alpha_{b+m c}F^\mu_{c+m a}
$$

by (9.10), because the other terms are zero as follows. The coefficient of $\omega^c$ is zero by the first equation in (5.7). The coefficient of $\omega^{d+m}$ is zero by the second equation in (5.7) and (8.37). The coefficient of $\omega^\beta$ is zero by the third equation of (5.7) and the third equation of (5.6). The coefficient of $\omega^\mu$ is zero by the third equation in (5.11) and (8.39).
Finally, the coefficient of \( e_\mu \) in \((dQ_a)e_{b+m}\) in (9.23) is, using (4.18) and (5.2)
\[
-F_{b+m+c}^\mu (\theta^c - L^c_a \omega^{d+m} + F_{c+m}^\alpha \omega^\alpha + F_{c+m}^\nu \omega^\nu) \\
+ (F_{b+m}^\alpha F_{ac}^\mu - F_{b+m+ac}^\mu + 2F_{a}^{\nu} F_{b+m+c}^\nu) \omega^c \\
+ (-L_{ab}^c F_{d+m+c}^\mu - L_{ad}^c F_{b+m}^\mu + F_{b+m}^\alpha F_{d+m}^\nu - F_{d+m}^\mu F_{b+m+a}^\mu) \omega^{d+m} \\
+ (-2L_{ab}^c F_{ac}^\mu - F_{b+m+ac}^\mu + F_{a+m}^\alpha F_{b+m+c}^\nu + F_{b+m+c}^\mu F_{c+m}^\nu) \omega^\alpha \\
+ (\delta_{ab} \delta_{\mu \nu} - F_{b+m+ac}^\nu - F_{a+m}^\alpha F_{b+m+ac}^\nu + F_{b+m+c}^\mu F_{c+m}^\nu - 4F_{aa}^\alpha F_{b+m}^\nu) \omega^\mu \\

= -F_{b+m+c}^\mu e^c_a
\]
by (9.10), because the other terms are zero as follows. The coefficient of \( \omega^c \) is zero by the first equation in (5.8). The coefficient of \( \omega^{d+m} \) is zero by the second equation in (5.8) and by (8.38). The coefficient of \( \omega^\alpha \) is zero by (5.11) and (8.39). The coefficient of \( \omega^\nu \) is zero by the fourth equation in (5.6).

That concludes the verification of (9.22).

The next case is to verify (9.12) when both sides are applied to \( e_\alpha \). We must verify that
\[
(d(Q_a e_\alpha) - Q_a d e_\alpha = (dQ_a) e_\alpha = \nu_a^b Q_0 e_\alpha + \nu_a^c Q_c e_\alpha \\
= \nu_a^a e_\alpha + F_{b+m+ac}^\alpha \nu_a^b e_c + F_{c+m}^\alpha \nu_{b+m}^b e_{c+m} - 2F_{a}^{\nu} F_{b+m}^\mu \nu_a^b e_{\mu}
\]
Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get
\[
(dQ_a) e_\alpha = (-F_{b+m+ac}^\alpha \omega^{b+m} + 2F_{aa}^\mu \omega^\mu - \theta_a^\alpha) e_\alpha \\
+ (-F_{a+m}^\alpha \omega^b + 2F_{aa}^\mu \omega^\mu - \theta_a^\alpha) e_0 \\
+ (dF_{a+m}^\alpha + F_{a+m}^\alpha \theta_a^b + F_{b+m}^\alpha \theta_a^b - F_{a+m}^\mu \theta_a^\mu \\
+ \delta_{ac} \omega^c - L_{ab}^c \theta_a^{b+m} - F_{a+m}^\mu \theta_a^\mu - F_{c+m}^\nu \theta_a^\nu) e_c \\
+ (F_{a+m}^\alpha \theta_a^c + dF_{a+m}^\alpha + F_{b+m}^\alpha \theta_a^{b+m} - 2F_{aa}^\mu \theta_a^\mu \\
- \delta_{ac} \omega^c + L_{ab}^c \theta_a^b - F_{a+m}^\nu \theta_a^\nu + F_{a+m}^\mu \theta_a^\mu) e_{c+m} \\
+ (F_{a+m}^\alpha \theta_a^b + F_{a+m}^\alpha \theta_a^{b+m} - 2F_{aa}^\mu \theta_a^\mu - F_{a+m}^\nu \theta_a^\nu \\
- F_{a+m}^\mu \theta_a^{b+m} + 2F_{a+m}^\mu \theta_a^\mu) e_\beta \\
+ (F_{a+m}^\alpha \theta_a^b + F_{a+m}^\alpha \theta_a^{b+m} - 2F_{aa}^\mu \theta_a^\mu - 2dF_{aa}^\mu \\
- F_{a+m}^\mu \theta_a^{b+m} + 2F_{a+m}^\mu \theta_a^\mu) e_\mu
\]
The coefficient of \( x \) is zero and the coefficient of \( e_0 \) is zero, both by (4.18). For the coefficient of \( e_c \), use (4.18), (5.2) and (8.4), and
add and subtract appropriate terms, to rewrite it as

$$F_{b+m,c}^\alpha (\theta_a^b - L_{ad}^b \omega^{d+m} + F_{b+m,a}^{\beta} + F_{b+m,a}^{\mu} \omega^\mu)$$

$$+ (F_{a+m,c}^\alpha - F_{b+m,c}^\alpha L_{ad}^b + F_{b+m,d}^c L_{ab}^c - F_{a+m,c}^\alpha F_{ad}^d) \omega^d$$

$$+ (F_{a+m,c}^\alpha d + 2F_{a+m,c}^\alpha F_{d+m,c}^d - F_{a+m,c}^\alpha F_{d+m}^d) \omega^{d+m}$$

$$+ (-F_{b+m,c}^a F_{b+m,a}^{\beta} - F_{b+m,a} F_{b+m,c}^{\beta} + 4F_{a+a}^\alpha F_{d+c}^\alpha + \delta a c \delta a b) \omega^\beta$$

$$+ (-F_{b+m,c}^a F_{b+m,a}^{\mu} + F_{b+m,a} F_{b+m,c}^{\mu} + F_{a+m,c}^\alpha - 2L_{ab}^c F_{b+m,a}^\alpha \omega^\mu$$

$$= F_{b+m,c}^a \omega^b$$

by (9.10), because the other terms are zero as follows. The coefficient of \(\omega^d\) is zero by the first equation of (5.7) and (8.37). The coefficient of \(\omega^{d+m}\) is zero by the second equation of (5.7). The coefficient of \(\omega^\beta\) is zero by the third equation of (5.7) and then the second equation of (5.6). The coefficient of \(\omega^\mu\) is zero by (5.11) and then (8.39).

The coefficient of \(c_{c+m}\) in \((dQ_a)c_\alpha\) in (9.25) is, using (4.18)

$$F_{c+m,b}^\alpha (\theta_a^b + L_{da}^b \omega^{d+m} + F_{b+m,a}^{\beta} + F_{b+m,a}^{\mu} \omega^\mu)$$

$$+ (F_{c+m,a}^\alpha d + 2F_{c+m,c}^\alpha F_{d+m,c}^d + F_{c+m,c}^\alpha F_{d+m}^d) \omega^d$$

$$+ (F_{c+m,a}^\alpha d + m + \chi_{a}^c F_{d+m,b}^d + F_{c+m,a}^\alpha F_{d+m}^d - L_{da}^b F_{c+m,b}^c) \omega^{d+m}$$

$$+ (F_{c+m,a}^\alpha b + F_{a+m,c}^\alpha F_{d+m,b}^b + 4F_{a+a}^\alpha F_{d+c}^\alpha + \delta a c \delta a b - F_{c+m,b}^a F_{b+m,a}^{\beta}) \omega^\beta$$

$$+ (F_{c+m,a}^\alpha b + F_{a+m,c}^\alpha F_{d+m,b}^b + 2L_{ab}^c F_{b+m,a}^\alpha \omega^\mu$$

$$= F_{c+m,a}^b \omega^a$$

by (9.10), because the other terms are zero as follows. The coefficient of \(\omega^d\) is zero by the first equation in (5.7). The coefficient of \(\omega^{d+m}\) is zero by the second equation in (5.7) and then (8.37). The coefficient of \(\omega^\beta\) is zero by the third equation in (5.7) and then the second equation in (5.6). The coefficient of \(\omega^\mu\) is zero by (5.11) and then (8.39).

The coefficient of \(c_\beta\) in \((dQ_a)c_\alpha\) in (9.25) is, using (4.18)

$$(F_{b+m,a} F_{b+m,c}^{\beta} + 2F_{a+a} F_{d+m,c}^{\beta} + 2F_{a+a} F_{d+m,c}^{\beta} + F_{b+m,a} F_{b+m,c}^{\beta} \omega^c$$

$$+ (F_{a+m,c} F_{d+m,c}^{\beta} + 2F_{a+a} F_{d+m,c}^{\beta} + 2F_{a+a} F_{d+m,c}^{\beta} + F_{a+m,c} F_{b+m,b}^{\alpha} \omega^{c+m}$$

$$- 2(F_{a+m,a} F_{b+m,a} F_{b+m,b}^{\alpha} + F_{a+m,a} F_{b+m,b}^{\alpha} + F_{a+m,a} F_{b+m,b}^{\alpha} + F_{a+m,a} F_{b+m,b}^{\alpha} \omega^\mu$$

$$= \delta a \beta (\omega^a + \omega^{a+m}) = \delta a \beta \omega^a$$

by (9.8), because the coefficient of \(\omega^c\) is \(\delta a \beta \delta a \beta\) by the second equation of (5.6), and the coefficient of \(\omega^{c+m}\) is also \(\delta a \beta \delta a \beta\) by (8.1), (8.2) and the second equation of (5.6); and the coefficient of \(\omega^\mu\) is zero by (8.1) and (8.2).
The coefficient of $e_\mu$ in $(dQ_a)e_\alpha$ in (9.25) is, using (4.18) and (5.2),
\begin{align*}
&-2F^\mu_{ab}(\theta^b_a + L^b_{ca}\omega^{c+m} + F^\beta_{b+m}\omega^\beta + F^\nu_{b+m}\omega^\nu) \\
&+ (-2F^\mu_{aac} + F^\alpha_{b+m}F^\mu_{b+m} - F^\mu_{b+m}F^\alpha_{b+m})\omega^c \\
&+ (-2F^\mu_{aa} + 2F^\mu_{ab}L^b_{ca} - F^\alpha_{a+m}F^\mu_{c+m} + F^\alpha_{a+m}F^\nu_{c+m})\omega^{c+m} \\
&+ 2(-F^\mu_{aa} + F^\beta_{b+m} - F^\alpha_{a+m}F^\beta_{b+m} + F^\nu_{a+m}F^\nu_{b+m})\omega^\beta \\
&+ 2(-F^\mu_{aa} + F^\mu_{ab}F^\nu_{b+m} - F^\mu_{a+m}F^\nu_{ab} + F^\mu_{b+m}F^\nu_{a+b+m})\omega^\nu \\
&= -2F^\mu_{ab}\nu^a_b
\end{align*}
by (9.8), because the coefficient of $\omega^c$ is zero by the first equation in (5.9), the coefficient of $\omega^{c+m}$ is zero by (5.11) and then (8.39), the coefficient of $\omega^\beta$ is zero by (4.18), and the second equation in (5.9); and the coefficient of $\omega^\nu$ is zero by (8.1), (8.3) and the third equation in (5.9).

This completes the verification of (9.24), which verifies that (9.12) holds when both sides are applied to $e_\alpha$.

The final case is to verify (9.12) when both sides are applied to $e_\mu$.

We must verify that
\begin{equation}
(dQ_a)e_\mu - Q_a(de_\mu) = (dQ_a)e_\mu = \nu^0_aQ_0e_\mu + \nu^b_aQ_be_\mu \\
= -\nu^a_\mu + F^\mu_{b+m}\nu^b_a = F^\mu_{a+m}\nu^b_a + F^\mu_{c+m}\nu^b_a = -2F^\mu_{ab}\nu^a_b
\end{equation}
Using (9.4) and (4.20), and gathering together the coefficients of each basis vector, we get for the left hand side
\begin{equation}
(dQ_a)e_\mu = (F^\mu_{b+m}\omega^{b+m} + 2F^\mu_{a+}\omega^\alpha - \theta^a_\mu)x \\
- (F^\mu_{a+m}\omega^{b} + 2F^\mu_{a+}\omega^\alpha + \theta^a_\mu)e_0 \\
+ (dF^\mu_{a+m}\theta^c - F^\mu_{b+m}\theta^c - 2F^\mu_{a\alpha}\theta^c + \delta_{a\alpha}\omega^\mu \\
- L^\mu_{a+}\theta^b_\mu - F^\mu_{c+m}\theta^c - F^\mu_{a+m}\theta^c_\mu)e_0 \\
+ (F^\mu_{a+m}\theta^c - dF^\mu_{a+m}\theta^c - F^\mu_{b+m}\theta^c - 2F^\mu_{a\alpha}\theta^c + \delta_{a\alpha}\omega^\mu \\
+ L^\mu_{a\alpha}\theta^b_\mu - F^\mu_{c+m}\theta^c - F^\mu_{a+m}\theta^c_\mu)e_0 \\
+ (F^\mu_{a+m}\theta^c - dF^\mu_{a+m}\theta^c - 2F^\mu_{a\alpha}\theta^c - F^\mu_{b+m}\theta^c + \delta_{a\alpha}\omega^\mu \\
- F^\mu_{a+m}\theta^b_\mu + 2F^\mu_{a\alpha}\theta^b_\mu)e_0 \\
+ (F^\mu_{a+m}\theta^c - dF^\mu_{a+m}\theta^c - 2F^\mu_{a\alpha}\theta^c - F^\mu_{b+m}\theta^c + \delta_{a\alpha}\omega^\mu \\
+ F^\mu_{a+m}\theta^b_\mu + 2F^\mu_{a\alpha}\theta^b_\mu + F^\mu_{b+m}\theta^b_\mu + 2F^\mu_{a\alpha}\theta^b_\mu)e_0.
\end{equation}

The coefficient of $x$ is zero by (4.18). The coefficient of $e_0$ is zero by (4.18) and (8.1).
After applying (5.2) and (4.18) and adding and then subtracting some terms in the definition of $\nu_a^b$ in (9.10), we can rewrite the coefficient of $e_c$ as

$$F^\mu b_{c+m}(\theta^b_a + F^b_{da}\omega^{d+m} + F^c_{b+m a}\omega^d + F^\nu_{b+m a}\omega^\nu)$$

$$+ (F^\mu c_{a+m c} - F^\mu_{b+m}L^b_{a d} + L^c_{a b}F^\nu_{b+m} + F^\alpha_{a+m c}F^\nu_{a+m d})\omega^d$$

$$+ (F^\mu_{a+m c d+m} + 2F^\mu_{a a}F^\nu_{d+m c} + F^\alpha_{a+m c}F^\nu_{a+m d+m})\omega^{d+m}$$

$$+ (F^\mu_{a+m c a} + F^\mu_{b+m a}F^\alpha_{b+m c} + 2L^c_{a b}F^\nu_{a b+m} - F^\mu_{b+m c}F^\nu_{b+m a})\omega^a$$

$$+ (F^\mu_{a+m c u} - F^\nu_{b+m a} - 4F^\mu_{a a}F^\nu_{a c} + \delta_{a c}\delta_{\mu\nu} - F^\mu_{b+m c}F^\nu_{b+m a})\omega^\nu$$

$$= F^\mu_{b+m c} \nu_a^b$$

by (9.10) and the following. The coefficient of $\omega^d$ is zero by the first equation in (5.8) and then (8.38). The coefficient of $\omega^{d+m}$ is zero by (8.1) and (5.8). The coefficient of $\omega^a$ is zero by (5.11) and then (8.39). The coefficient of $\omega^\mu$ is zero by the third equation in (5.8) and then (8.1) and (8.3) and then the fourth equation in (5.6).

Using (5.2) and (4.18), we can rewrite the coefficient of $e_{c+m}$ in $(dQ_a)e_{\mu}$ in (9.27) as

$$- F^\mu c_{a+m b}(\theta^b_a + L^b_{d a}\omega^{d+m} + F^c_{b+m a}\omega^d + F^\nu_{b+m a}\omega^\nu)$$

$$+ (-F^\mu c_{e+m ab} + 2F^\mu_{a a}F^c_{c+m b} + F^\alpha_{c+m a}F^\nu_{a b})\omega^b$$

$$+ (L^b_{d a}F^\nu_{c+m b} - F^\nu c_{c+m a d+m} + L^b_{a b}F^\nu c_{d+m b} + F^\alpha_{c+m a}F^\nu_{a d+m})\omega^{d+m}$$

$$+ (F^\mu_{c+m b}F^\alpha_{b+m a} - F^\mu_{c+m a a} + F^\mu_{a+m b}F^\alpha_{c+m b} + 2F^\mu_{a b}L^c_{a b})\omega^a$$

$$+ (F^\mu_{c+m b}F^\nu_{c+m a} - F^\nu c_{c+m a b} - F^\mu_{a+m b}F^\nu_{c+m b} - 4F^\mu_{a a}F^\nu_{a c+m} + \delta_{a c}\delta_{\mu\nu})\omega^\nu$$

$$= - F^\mu c_{c+m b} \nu_a^b$$

by (9.10) and the following. The coefficient of $\omega^b$ is zero by the first equation in (5.8). The coefficient of $\omega^{d+m}$ is zero by the second equation in (5.8) and then (8.1) and (8.38). The coefficient of $\omega^a$ is zero by (5.11), then (8.2) and (8.3) and (8.39). The coefficient of $\omega^\nu$ is zero by the third equation in (5.8), then (8.1) and (8.3) and then the fourth equation in (5.6).
Using (5.2) and (4.18), we can rewrite the coefficient of $e_\alpha$ in $(dQ_a)e_\mu$ in (9.27) as

$$
-2F_{ab}^\mu (\theta_a^b + L_{ba}^b \omega^{c+m} + F_{b+m\alpha, a} \omega^\alpha + F_{b+m\alpha, a} \omega^{c+m})
+ (-2F_{ac}^\mu - F_{b+m\alpha, a} F_{b+m\alpha, c} + F_{b+m\alpha, a} F_{b+m\alpha, c}) \omega^c
+ (-2F_{ac}^\mu + 2F_{ab}^\mu L_{ac}^b + F_{a+b-m\alpha, a} F_{a+b-m\alpha, c} - F_{a+b-m\alpha, a} F_{a+b-m\alpha, c}) \omega^{c+m}
+ 2(-F_{ac}^\mu F_{b+m\alpha, a} + F_{bc}^\mu F_{a+b-m\alpha, c} + F_{bc}^\mu F_{a+b-m\alpha, c}) \omega^\beta
+ 2(-F_{ac}^\mu F_{b+m\alpha, a} + F_{bc}^\mu F_{a+b-m\alpha, c} + F_{bc}^\mu F_{a+b-m\alpha, c}) \omega^\beta
= -2F_{ab}^\mu \nu_a^b
$$

by (9.10) and the following. The coefficient of $\omega^c$ is zero by the first equation in (5.9). The coefficient of $\omega^{c+m}$ is zero by (5.11), then (8.2) and (8.3) and then (8.39). The coefficient of $\omega^\beta$ is zero by the second equation in (5.9) and then (8.1)-(8.3). The coefficient of $\omega^\nu$ is zero by the third equation in (5.9), then (8.1) and (8.3).

Using (4.18), we can rewrite the coefficient of $e_\mu$ in $(dQ_a)e_\mu$ in (9.27) as

$$
(-F_{b+m\alpha, a} F_{b+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c}) \omega^c
+ (-2F_{a+m\alpha, b} F_{a+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c}) \omega^{c+m}
+ 2(-F_{a+m\alpha, b} F_{a+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c} - 2F_{a+m\alpha, b} F_{a+m\alpha, c}) \omega^\alpha
= -\delta_{\mu\nu}(\omega^a + \omega^{a+m}) = -\delta_{\mu\nu} v^a
$$

by (8.1), (8.3) and the fourth equation in (5.6). This completes the verification of (9.12) when both sides are applied to $e_\mu$, and therefore also completes the verification of (9.12).

10. THE QUADRATIC FORMS

For the remainder of the paper, we will again refer to the two multiplicities as $m_1$ and $m_2$, rather than $m$ and $N$, respectively, and we will no longer use the Einstein summation convention. Our task now is to solve (8.1) through (8.4). It is known that $m_1 = m_2$ only when $m_1 = m_2 = 1$, which is of FKM-type, or $m_1 = m_2 = 2$, which is not of FKM-type [1]. Therefore we assume $m_1 \neq m_2$ henceforth. Our convention is that $m_1 < m_2$ and we denote by $M_+$ (respectively, $M_-$) the focal submanifold whose co-dimension is $m_1 + 1$ (respectively, $m_2 + 1$) in the ambient sphere. We change the Cartan-Münzner polynomial $F$ to $-F$ if necessary so that always $M_+ = f^{-1}(1)$ with respect to the isoparametric function $f$. In view of Theorem 24, we look for conditions on $m_1$ and $m_2$ that imply the validity of (8.1) and the spanning property.
As in Section 4 let $\mathbf{x} \in M_+$ and let $e_0$ be a unit normal vector to $M_+$ at $\mathbf{x}$ for which the shape operator $S_{e_0}$ assumes the eigenspaces $V_0, V_+$ and $V_-$ with eigenvalues 0, 1, and $-1$, respectively. For an orthonormal basis $e_0, \ldots, e_{m_1}$ of the normal space to $M_+ \setminus \mathbf{x}$ we introduce the quadratic homogeneous polynomials

$$\tilde{p}_i(\mathbf{y}) := S_{e_i} \mathbf{y} \cdot \mathbf{y},$$

for $0 \leq i \leq m_1$, where $\mathbf{y}$ is tangent to $M_+$ at $\mathbf{x}$. When such $\mathbf{y}$ has no $V_0$ component, we shall write $\mathbf{z}$ instead of $\mathbf{y}$. Regarding $V_+ \oplus V_-$ as a subspace of $\mathbb{R}^{2l}$ by parallel translation, consider the set

$$\mathcal{D} := \{ \mathbf{z} \in V_+ \oplus V_- : |\mathbf{z}| = 1, \tilde{p}_i(\mathbf{z}) = 0, 0 \leq i \leq m_1 \}.$$

**Proposition 25.** $\mathcal{D} = (V_+ \oplus V_-) \cap M_+$.

**Proof.** This follows from the formula of [25, I, pp.524-526], that reads

$$F(t\mathbf{x} + \mathbf{y} + \mathbf{w}) = t^4 + (2|\mathbf{y}|^2 - 6|\mathbf{w}|^2)t^2 + 8\left(\sum_{i=0}^{m_1} \tilde{p}_i(\mathbf{y})w_i\right)t$$

$$+ |\mathbf{y}|^4 - 2\sum_{i=0}^{m_1} (\tilde{p}_i(\mathbf{y}))^2 + 8\sum_{i=0}^{m_1} q_i(\mathbf{y})w_i$$

$$+ 2\sum_{i,j=0}^{m_1} (\nabla \tilde{p}_i \cdot \nabla \tilde{p}_j)w_iw_j - 6|\mathbf{y}|^2 |\mathbf{w}|^2 + |\mathbf{w}|^4. \tag{10.1}$$

Here, the homogeneous polynomials of degree three, $q_i(\mathbf{y})$, are the components of the third fundamental form of $M_+$, $\mathbf{w} = \sum_{i=0}^{m_1} w_i e_i$, and $\mathbf{y}$ is tangent to $M_+$. For the convenience of the reader, let us briefly recall that Ozeki and Takeuchi expanded $F(t\mathbf{x} + \mathbf{y} + \mathbf{w})$ in terms of $t$ and substituted it into its governing partial differential equations mentioned in Section 2 to get

$$F(t\mathbf{x} + \mathbf{y} + \mathbf{w}) = t^4 + At^2 + Bt + C,$$

where $A$ is derived on p. 525, $B$ is on p. 526, and $C = C_0 + \cdots + C_4$, in which $C_s$, given on p. 526, is the homogeneous part of $C$ of degree $s$ in the normal coordinates $w_0, \ldots, w_{m_1}$. When one sets $t = 0$, $\mathbf{w} = 0$ and $\mathbf{y} = \mathbf{z} \in V_+ \oplus V_-$ in the formula (10.1) one gets

$$F(\mathbf{z}) - |\mathbf{z}|^4 = -2\sum_{i=0}^{m_1} (\tilde{p}_i(\mathbf{z}))^2.$$

Hence when $|\mathbf{z}| = 1$, we have $F(\mathbf{z}) = 1$ if and only if $\tilde{p}_i(\mathbf{z}) = 0$ for $0 \leq i \leq m_1$. \qed
Remark 26. It is not obvious that the set $\mathcal{D}$ is non-empty. In Theorem 47 of Section 12, we will prove that $\mathcal{D}$ is non-empty when $m_2 \geq 2m_1 - 1$. Proposition 28 below still holds in the case where $\mathcal{D}$ is empty. In that case, the zero locus of each set of polynomials in Proposition 28 is empty.

In view of Proposition 25 we set $p_i$ to be the restriction of $\frac{1}{2}\bar{p}_i$ to the space $V_+ \oplus V_-$ for $1 \leq i \leq m_1$, and set $p_0$ to be the restriction of $\bar{p}_0$ to this space. These are the quadratic polynomials $p_0, p_a$ defined in (6.6). Recall from (4.26) and (6.6), that relative to a second order Darboux frame we have variables $x = (x_\alpha)$ and $y = (y_\mu)$ in terms of which these polynomials are

$$p_0(x, y) = \sum_{\alpha=1}^{m_2} (x_\alpha)^2 - \sum_{\mu=1}^{m_2} (y_\mu)^2, \quad p_a(x, y) = \sum_{\alpha, \mu=1}^{m_2} F_{\alpha\mu} x_\alpha y_\mu$$

For notational ease, as the context should remove any possibility of confusion, we will stick to the range $1 \leq \alpha, \mu \leq m_2$ for $x_\alpha$ and $y_\mu$ from now on even though $\alpha$ and $\mu$ live in the designated ranges as given in (4.6).

As mentioned in Section 4, we know $e_0$ also lies in $M_+$ with the normal space $\text{span}(x, e_{m_1+1}, \ldots, e_{2m_1})$. The $0, +1, -1$ eigenspaces of the shape operator $S_x$ at $e_0$ are, respectively, $\text{span}(e_1, \ldots, e_{m_1})$, $V_+$ and $V_-$. With respect to the normal basis $x, e_p, m_1 + 1 \leq p \leq 2m_1$, at $e_0$, we let $\bar{p}_0, \ldots, \bar{p}_{m_1}$ be the counterparts of $p_0, \ldots, p_{m_1}$, respectively, as in (6.9). Then Proposition 25 immediately gives the following simple but crucial observation.

Proposition 27. $\mathcal{D} = \{ z \in V_+ \oplus V_- : |z| = 1, \bar{p}_i(z) = 0, 0 \leq i \leq m_1 \}.$

Now $\mathcal{D}$ can be viewed from a different angle. Observe that all $z = (x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}) \in \mathcal{D}$ must satisfy $\sum_{\alpha=1}^{m_2} (x_\alpha)^2 + \sum_{\mu=1}^{m_2} (y_\mu)^2 = 1$. It follows that $z \in S^{m_2-1} \times S^{m_2-1}$ due to the fact that $p_0(z) = 0$, where $S^{m_2-1}$ is the standard sphere of radius $1/\sqrt{2}$. The real projective variety out of $S^{m_2-1} \times S^{m_2-1}$ is $\mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1}$. Note that the solution to $p_a = 0, 1 \leq a \leq m_1$, lives naturally in $\mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1}$, which is parametrized by $[x_1 : \ldots : x_{m_2}] \times [y_1 : \ldots : y_{m_2}]$. As a consequence the projectivized $\tilde{\mathcal{D}}$ in $\mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1}$ via the map $S^{m_2-1} \times S^{m_2-1} \longrightarrow \mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1}$ is exactly

$$\mathcal{D} := \{ [z] \in \mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1} : p_a(z) = 0, 0 \leq a \leq m_1 \}.$$

Note that $\mathcal{D} \neq \emptyset$ if and only if $\mathcal{D} \neq \emptyset$. Since the $+1$ and $-1$ eigenspaces of the shape operator $S_x$ at $e_0$ are $V_+$ and $V_-$, respectively, it follows
from (10.2) that \( \mathfrak{p}_0 = p_0 \). Hence, Proposition 27 can be rephrased as follows.

**Proposition 28.** The zero locus of \( p_1, \ldots, p_{m_1} \) in \( \mathbb{R}P^{m_2-1} \times \mathbb{R}P^{m_2-1} \) is identical with that of \( \mathfrak{p}_1, \ldots, \mathfrak{p}_{m_1} \).

**Lemma 29.** If \( m_2 \geq m_1 + 2 \), then the quadratic forms \( p_1, \ldots, p_{m_1} \) are linearly independent and irreducible, both over the real numbers \( \mathbb{R} \) and over the complex numbers \( \mathbb{C} \).

**Proof.** The quadratic form \( p_a(x, y) \) is given by

\[
4p_a(x, y) = \begin{pmatrix} 0 & A_a \\ tA_a & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2A_a y \cdot x
\]

where \( A_a \) is the matrix with respect to \( e_\alpha, e_\mu \) of the operator defined in the first equation of (6.3). Recall that the rank of \( p_a \) is defined to be the rank of the matrix of the associated bilinear form. Hence \( \text{rank}(p_a) = 2 \text{rank}(A_a) \).

Let \( S_a = U + V \) be the shape operator given in (6.5), where \( U \) is the matrix that retains the \( A_a \) and \( tA_a \) blocks and is zero elsewhere. Then \( \text{rank} S_a \leq \text{rank} U + \text{rank} V \). Since \( \text{rank} S_a = 2m_2 \), \( \text{rank} U = 2 \text{rank} A_a \) and \( \text{rank} V \leq 2m_1 \), we get

\[
2(m_2 - m_1) \leq 2 \text{rank} A_a = \text{rank}(p_a)
\]

for all \( a \), as proved by Ozeki and Takeuchi [25, II, p45].

If \( p_a \) is reducible, then \( p_a = fg \) is a product of linear forms \( f = a_\alpha x_\alpha + a_\mu y_\mu \) and \( g = b_\alpha x_\alpha + b_\mu y_\mu \). If we let \( a = t(a_\alpha a_\mu) \) and \( b = t(b_\alpha b_\mu) \) \( \in \mathbb{R}^{2m_2} \) then the symmetric matrix of the quadratic form \( 4p_a(x, y) \) must be \( (a'b + b'a)/2 \), which has rank \( \leq 2 \), as each column is a linear combination of \( a \) and \( b \). In particular, if \( m_2 - m_1 \geq 2 \), then \( \text{rank}(p_a) \geq 4 > 2 \) and hence, \( p_a \) is irreducible over \( \mathbb{R} \). Notice that this discussion is unchanged if we work over the complex numbers, which shows that they are irreducible over \( \mathbb{C} \) as well. Linear independence of \( p_1, \ldots, p_{m_1} \) over \( \mathbb{R} \) is equivalent to linear independence of \( A_1, \ldots, A_{m_1} \), which follows under our hypotheses from Proposition 7. Being real polynomials, they are also linearly independent over \( \mathbb{C} \).

\[ \square \]

11. Some commutative algebra and algebraic geometry

We will explore in more depth the fact that \( p_a, 1 \leq a \leq m_1 \), are irreducible when \( m_2 \geq m_1 + 2 \) and are bihomogeneous, i.e., are homogeneous in \( x_1, \ldots, x_{m_2} \) and in \( y_1, \ldots, y_{m_2} \), of bi-degree \( (1, 1) \) in this section. We shall pursue commutative algebra only to the extent that serves our need, and shall stress the geometry behind the algebra. A few ad hoc proofs and examples will be given to convey to the reader,
who might be unfamiliar with the subject, some intuition about the concepts encountered. Henceforth, $n$ is just an index that has nothing to do with the dimension of the ambient sphere in which the isoparametric hypersurface sits.

**Definition 30.** Let $F$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $F[x_1, \ldots, x_s, y_1, \ldots, y_s]$ be the polynomial ring in variables $x_1, \ldots, x_s, y_1, \ldots, y_s$ over $F$. Given bihomogeneous polynomials $p_1, \ldots, p_n$, we say that the ideal $I := (p_1, \ldots, p_n)$ in $F[x_1, \ldots, x_s, y_1, \ldots, y_s]$ is reduced if

(i). The bi-projective variety $P_bV_I := \{(x, y) \in FP_{s-1} \times FP_{s-1} : p_a(x, y) = 0, 1 \leq a \leq n\}$ is not empty, and

(ii). whenever $f \in F[x_1, \ldots, x_s, y_1, \ldots, y_s]$ satisfies $f|_{P_bV_I} \equiv 0$ then we have $f = p_1f_1 + \cdots + p_nf_n$ for some $f_1, \ldots, f_n \in F[x_1, \ldots, x_s, y_1, \ldots, y_s]$.

We call the affine variety $V_I := \{(x, y) \in C^s \times C^s : p_a(x, y) = 0, 1 \leq a \leq n\}$ a bi-affine cone.

For instance, when $F = \mathbb{C}$, the radical of $I$, denoted by $\text{rad}(I)$, is always reduced, if $P_bV_I \neq \emptyset$. This is Hilbert’s Nullstellensatz indeed [11]. In particular, since a prime ideal equals its radical, the ideal $I$ will be reduced if $I$ is a prime ideal. $P_bV_I$ is not empty automatically in this case, because otherwise $V_I = (\{0\} \times C^s) \cup (C^s \times \{0\})$ would not be irreducible. We will extensively probe the primeness of $I$ subsequently. (See [14] and [21] for bi-projective geometry.) Before we proceed, let us introduce a notation. When $p$ is a real polynomial, we denote by $p^C$ the same polynomial whose variables are over the complex numbers. We call $p^C$ the complexification of $p$. Likewise, when $p_1, \cdots, p_n$ are bihomogeneous in $R[x_1, \ldots, x_s, y_1, \ldots, y_s]$, we denote by $V$ the resulting real bi-affine cone and by $V^C$ the complex bi-affine cone defined by the complexifications of $p_1, \cdots, p_n$.

**Lemma 31.** Suppose $V$ is a bi-affine cone in $R^s \times R^s$ defined by the real polynomials $p_1, \cdots, p_n$, such that its complex counterpart $V^C$ is irreducible and such that $\dim_R(V) = \dim_C(V^C)$. If a real polynomial $p(x_1, \ldots, x_s, y_1, \ldots, y_s)$ satisfies $p|_V \equiv 0$, then $p^C|_{V^C} \equiv 0$. Here, by the dimension of $V$ we mean the maximal dimension of all the irreducible components of $V$.

**Proof.** Suppose $p^C|_{V^C}$ is not identically zero on $V^C$. Then $p^C$ cuts out a subvariety $X$, all of whose irreducible components are of codimension
1 in \( V^C \) [28, p59]. Clearly, \( V \subseteq X \). Then we have
\[
\dim_{\mathbb{R}}(V) \leq \dim_{\mathbb{C}}(X) = \dim_{\mathbb{C}}(V^C) - 1,
\]
in contradiction to the assumption that \( \dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V^C) \). The inequality holds true because any real analytic parametrization \( \sigma : t = (t_1, \ldots, t_k) \mapsto (x_1, \ldots, x_s, y_1, \ldots, y_s) \in V \) around a smooth point, at \( t = 0 \), of \( V \), satisfies \( p_1(\sigma(t)) = \cdots = p_n(\sigma(t)) = p(\sigma(t)) = 0 \). The convergent power series defining \( \sigma \) remain so when \( t_1, \ldots, t_k \) are allowed to be complex variables, and then \( \sigma(t) \) is a holomorphic map, nonsingular at \( t = 0 \), such that \( p_1^C(\sigma(t)) = \cdots = p_n^C(\sigma(t)) = p^C(\sigma(t)) = 0 \) because a holomorphic function vanishing on the real part is identically zero. That is, \( \sigma(t) \), with \( t \) complex, is a holomorphic map, nonsingular at \( t = 0 \), into \( X \). Therefore, we conclude that \( \dim_{\mathbb{C}}(X) \geq \dim_{\mathbb{R}}(V) \). \( \square \)

**Proposition 32.** If \( p_1, \ldots, p_n \in \mathbb{R}[x_1, \ldots, x_s, y_1, \ldots, y_s] \) are bihomogeneous polynomials of positive degree in each set of variables, and if \( p_1^C, \ldots, p_n^C \), their complexifications, are such that
\[
\begin{align*}
(1): V^C := \{ z \in \mathbb{C}^s \times \mathbb{C}^s : p_a^C(z) = 0, 1 \leq a \leq n \} & \text{ is irreducible}, \\
(2): \text{rad}(I) = I, \text{ where } I := (p_1^C, \ldots, p_n^C), & \text{ and} \\
(3): \dim_{\mathbb{R}}(V) = \dim_{\mathbb{C}}(V^C), \text{ where } V := \{ z \in \mathbb{R}^s \times \mathbb{R}^s : p_a(z) = 0, 1 \leq a \leq n \}.
\end{align*}
\]
then the real ideal \( (p_1, \ldots, p_n) \) is reduced.

**Proof.** \( I \) is a prime ideal by the first two assumptions. Therefore, the remark immediately after Definition 30 ensures that \( \mathbb{P}_bV^C \) is not empty. Moreover, \( \dim_{\mathbb{C}}(V^C) > s \) by the first assumption and the fact that the reducible \( (\mathbb{C}^s \times \{0\}) \cup (\{0\} \times \mathbb{C}^s) \) is contained in \( V^C \). Hence \( \mathbb{P}_bV \) is not empty either by the third assumption. So the first condition in Definition 30 holds. Let \( f \) be a real polynomial vanishing on \( \mathbb{P}_bV \) so that \( f \) vanishes on \( V \) as well; by Lemma 31 its complexification \( f^C \) vanishes on \( V^C \). It follows from the reducedness of \( I \) that there are complex bi-homogeneous polynomials \( h_1, \ldots, h_n \) such that
\[
f^C = p_1^C h_1 + \cdots + p_n^C h_n.
\]
Let \( f_1, \ldots, f_n \) be, respectively, the real parts of \( h_1, \ldots, h_n \) when they are restricted to the real variables. We have, by the realness of \( f \) and \( p_1, \ldots, p_n \), that \( f = p_1 f_1 + \cdots + p_n f_n \). \( \square \)

We now review some important notions and properties from commutative algebra, leaving detailed expositions to [11] and [19].

**Definition 33.** Let \( R \) be a commutative ring with identity. We say that \( n \) elements \( x_1, \ldots, x_n \in R \) form a regular sequence if \( (x_1, \ldots, x_n) \neq \)
$R$, $x_1$ is not a zero divisor in $R$ and $x_{i+1}$ is not a zero divisor in the quotient ring $R/I_i$, where $I_i$ is the ideal $(x_1, \ldots, x_i)$, for $1 \leq i \leq n - 1$.

**Example 34.** A single nonconstant $p \in \mathbb{C}[z_1, \ldots, z_L]$ clearly forms a regular sequence.

**Example 35.** Let $p_1$ and $p_2$ in $\mathbb{C}[z_1, \ldots, z_L]$ be relatively prime homogeneous polynomials of degree $\geq 1$. Then $p_1$ and $p_2$ form a regular sequence. This follows simply from the fact that $p_2f = p_1g$ implies $f = p_1h$ for some $h$. Moreover, $(p_1, p_2)$ is not the entire polynomial ring due to the homogeneity of $p_1$ and $p_2$.

**Definition 36.** Let $\mathcal{P}$ be a prime ideal in a commutative ring $R$ with identity. We define the codimension of $\mathcal{P}$ to be

$$\text{codim}(\mathcal{P}) = \sup\{s : \text{there is a prime chain } \mathcal{P}_s \subset \cdots \subset \mathcal{P}_1 \subset \mathcal{P}_0 = \mathcal{P} \},$$

where the set inclusions are all proper. For an arbitrary ideal $I$ we define

$$\text{codim}(I) = \inf_{i \in \mathcal{P}} \{\text{codim}(\mathcal{P})\},$$

and define the depth of $I$ to be

$$\text{depth}(I) = \sup\{n : \text{there is a regular sequence } x_1, \ldots, x_n \in I \}.$$

We define the dimension of $R$ to be

$$\text{dim}(R) = \sup\{s : \text{there is a prime chain } \mathcal{P}_s \subset \cdots \subset \mathcal{P}_1 \subset \mathcal{P}_0 \subset R \}.$$

Lastly, $R$ is Cohen-Macaulay if, for every maximal ideal $\mathcal{M}$ of $R$ (and such ideals are necessarily prime), we have

$$\text{depth}(\mathcal{M}) = \text{codim}(\mathcal{M}).$$

**Example 37.** Consider $R := \mathbb{C}[x, y, z]$ with $p_1 = xz$ and $p_2 = yz$. The ideal $I := (p_1, p_2)$ has the property $\text{rad}(I) = I$ so that $R/I$ is the coordinate ring of the zero locus of $p_1$ and $p_2$, which is made up of the $(x, y)$-plane and the $z$-axis. It is not hard to see that $\text{dim}(R/I) = 2 \neq 1$, the ambient dimension minus the number of equations. So the ring $R/I$ is not Cohen-Macaulay. In fact, at the origin the maximal ideal $\mathcal{M} = (x, y, z)/I$ is the first term in a maximal descending prime chain $(x, y, z)/I, (y, z)/I$ and $(z)/I$ so that $\text{codim}(\mathcal{M}) = 2$. However, $\text{depth}(\mathcal{M}) = 1$, since $x + z \mod(I)$, for instance, forms a maximal regular sequence in $\mathcal{M}$.

The following ingredient, on the other hand, generates many Cohen-Macaulay rings.
FACT([11, p455]). If \( p_1, \ldots, p_n \) form a regular sequence in the ring \( R := C[z_1, \ldots, z_L] \) with ideal \( I = (p_1, \ldots, p_n) \), then \( \text{codim}(I) = n \), the ring \( R/I \) is Cohen-Macaulay, and \( \dim(R/I) = L - n \).

**Remark 38.** The FACT can be interpreted geometrically. In the case when \( \text{rad}(I) = I \), for instance, the quotient ring \( R/I \) is the coordinate ring of an affine variety. This quotient ring being Cohen-Macaulay says that each point of the affine variety is the zero locus of \( L \) coordinate functions from \( R/I \) (technically, in a maximal regular sequence vanishing at the point), and thus the codimension in the variety of each point is the expected value \( L - n \). The affine variety is then called a complete intersection. It is of dimension \( L - n \) on all of its irreducible components.

We now come to the major recipe for inductively constructing Cohen-Macaulay rings in this paper.

**Proposition 39.** If \( p_1, \ldots, p_n \) are linearly independent homogeneous polynomials of equal degree \( \geq 1 \) in the ring \( C[z_1, \ldots, z_L] \) such that the ideal \( (p_1, \ldots, p_{n-1}) \) is prime and such that \( p_1, \ldots, p_{n-1} \) form a regular sequence, then \( p_1, \ldots, p_n \) form a regular sequence. In particular, the FACT above implies that the quotient ring \( C[z_1, \ldots, z_L]/(p_1, \ldots, p_n) \) is Cohen-Macaulay.

**Proof.** We know \( V_{n-1} \) is irreducible since \( I_{n-1} := (p_1, \ldots, p_{n-1}) \) is prime. Thus \( p_n \) cannot vanish identically on \( V_{n-1} \). Otherwise the Nullstellensatz applied to \( p_n \) on the prime \( I_{n-1} \) would imply

\[
p_n = p_1 f_1 + \cdots + p_{n-1} f_{n-1}
\]

for some \( f_1, \ldots, f_{n-1} \in C[z_1, \ldots, z_L] \). As shown in Proposition 11, we may assume that \( f_1, \ldots, f_{n-1} \) are constant polynomials, because all of \( p_1, \ldots, p_n \) are homogeneous of the same degree \( \geq 1 \). But this would imply that \( p_1, \ldots, p_n \) were linearly dependent, which is not the case by assumption.

Suppose there are \( f, f_1, \ldots, f_{n-1} \in C[z_1, \ldots, z_L] \) such that

\[
p_n f = p_1 f_1 + \cdots + p_{n-1} f_{n-1}.
\]

Then \( f|_{V_{n-1}} \equiv 0 \) since \( p_n \) does not vanish identically on the irreducible \( V_{n-1} \). So once more the Nullstellensatz applied to \( f \) on \( I_{n-1} \) implies that

\[
f = p_1 g_1 + \cdots + p_{n-1} g_{n-1}
\]

for some \( g_1, \ldots, g_{n-1} \in C[z_1, \ldots, z_L] \).

Lastly, \( (p_1, \ldots, p_n) \neq C[z_1, \ldots, z_L] \) since \( p_1, \ldots, p_n \) are all homogeneous of the same degree \( \geq 1 \). This confirms that \( p_1, \ldots, p_n \) form a regular sequence. \( \square \)
For our later applications on the variety level, Proposition 39 is not quite sufficient, because the ring $\mathbb{C}[z_1, \ldots, z_L]/(p_1, \ldots, p_n)$ in the proposition, though being Cohen-Macaulay, may have nilpotent elements, in which case the ring is not the coordinate ring of an affine variety. If the ring contains no nilpotent elements, then it is called reduced.

**Example 40.** Let $p_1 = y - x^2$ and $p_2 = y$ in $\mathbb{C}[x, y]$. The zero locus of $p_1$ and $p_2$ is $\{(0, 0)\}$. However, the Cohen-Macaulay ring $\mathbb{C}[x, y]/(p_1, p_2)$ has a nilpotent, namely, $x \mod((p_1, p_2))$. Geometrically, the parabola $y = x^2$ intersects $y = 0$ with multiplicity 2.

What we must do now is to find conditions under which the quotient ring in Proposition 39 is reduced, in which case the variety associated with the ring is called a Cohen-Macaulay variety.

**Proposition 41.** Let $J_n$ be the subvariety of the variety $V_n := \{z \in \mathbb{C}^L : p_1(z) = 0, \ldots, p_n(z) = 0\}$ where the Jacobian matrix of $p_1, \ldots, p_n$ is not of rank $n$. If $\text{codim}(J_n) \geq 1$ in $V_n$, then the affine coordinate ring $\mathbb{C}[z_1, \ldots, z_L]/(p_1, \ldots, p_n)$ is reduced.

**Proof.** This is just Serre’s criterion of reducedness [11, p457].

**Remark 42.** If we assume in Proposition 39 that $J_{n-1}$, the subvariety of $V_{n-1} = \{z : p_1(z) = \cdots = p_{n-1}(z) = 0\}$ where the Jacobian of $p_1, \ldots, p_{n-1}$ is not of rank $n-1$, is of codimension $\geq 2$ in $V_{n-1}$, then we can give a somewhat more geometric account of Proposition 41 as follows. (In fact, in our applications to follow, $\text{codim}(J_{n-1}) \geq 2$ always holds true.) Let $R = \mathbb{C}[z_1, \ldots, z_L]$, let $I = (p_1, \ldots, p_{n-1})$ and let $J = (p_n)$. We must show $R/(I + J)$ has no nilpotents. That is, whenever $f \in R$ satisfies

$$f^k = p_1 f_1 + \cdots + p_n f_n \in I + J$$

for some $k$ and $f_1, \ldots, f_n$, we must have $f \in I + J$. We may assume $f^k$ is not in $I$, or else we are done since then $f \in I$ by the primeness of $I$. It follows that $f$ is nonzero on $V_{n-1}$ and is zero on $V_n$.

Let $V_n = W_1 \cup \cdots \cup W_s$ be the irreducible decomposition of $V_n$ in $V_{n-1}$. We know $\text{codim}(W_i) = 1$ in $V_{n-1}$ for all $i$. Then by $\text{codim}(J_n) \geq 1$ in $V_n$, the polynomial $p_n$ cuts out $W_i$ with multiplicity 1 for each $i$ (it comes down to the implicit function theorem in calculus). That is, $p_n = 0$ defines the divisor $W_1 + \cdots + W_s$ in $V_{n-1}$.

Now since $f$ vanishes on $V_n$, the divisor defined by $f = 0$ assumes multiplicity $\geq 1$ on each $W_i$. At this point the principle that says that the poles get cancelled by the zeros seems to suggest that the rational function $f/p_n$ is regular everywhere on $V_{n-1}$. This is certainly true if $V_{n-1}$ is smooth [28, p129], because the germs of local regular functions...
on $V_{n-1}$ then form a unique factorization domain; more generally, the normality of the variety suffices for the conclusion [28, p111]. From this it follows that $(f/p_n)|_{V_{n-1}} = g$ for some regular $g$ on $V_{n-1}$. In other words, $(f - p_ng)|_{V_{n-1}} \equiv 0$. Therefore,

$$f - p_ng = p_1g_1 + \cdots + p_{n-1}g_{n-1} \in I$$

by the primeness of $I$. We conclude that $f \in I + J$, proving the reducedness of $R/(I + J)$.

It remains to ensure the normality of $V_{n-1}$, which is true if the codimension of $J_{n-1}$ is at least 2. This is a consequence of Serre’s criterion of primeness [11, p457], because $V_{n-1}$ is a Cohen-Macaulay variety due to $\text{codim}(I) = n - 1$. In any event we resort to Serre’s criterion one way or another.

The next proposition plays a vital role in the applications to follow.

**Proposition 43.** We assume the notation in Proposition 41. If furthermore $\text{codim}(J_n) \geq 2$ in $V_n$ and $V_n$ is connected, then $(p_1, \ldots, p_n)$ is a prime ideal.

**Proof.** Proposition 41 asserts that $V_n$ is a connected Cohen-Macaulay variety. Now $X_n$, the complement of $J_n$ in $V_n$, is smooth on the one hand. On the other hand, $X_n$ is also connected on account of Hartshorne’s connectedness theorem [11, p454], that says that a connected Cohen-Macaulay variety remains connected when a subvariety of codimension $\geq 2$ is removed. Being both smooth and connected, $X_n$ must be irreducible. However, since $\text{codim}(J_n) \geq 2$, $J_n$ cannot be an irreducible component of $V_n$ due to the fact that a Cohen-Macaulay variety is of equal dimension on all of its irreducible components. $V_n$ is then irreducible. As a consequence $(p_1, \ldots, p_n)$ is a prime ideal because Proposition 41 establishes the reducedness of $(p_1, \ldots, p_n)$. □

**Example 44.** This example shows that $\text{codim}(J_n) \geq 2$ in $V_n$ is a must in Proposition 43. Let $p_1 = z$ and $p_2 = x^2 - y^2 + z^2$ in $C[x, y, z]$. Then $V_2 = \{(x, \pm x, 0)\}$ and $J_2 = \{(0, 0, 0)\}$, which is of codimension 1 in $V_2$. But $V_2$ is reducible albeit connected. It also illustrates that the codimension 2 condition in Hartshorne’s connectedness theorem cannot be improved to codimension 1.

12. THE CLASSIFICATION THEOREM

We now return to the isoparametric case. For a given second order Darboux frame field (4.16) along $x$ on $U \subset M$, recall that we have, for
1 ≤ a ≤ m₁, bihomogeneous polynomials
\[ p_a = \sum_{\alpha, \mu=1}^{m_2} F^\mu_{\alpha a} x_\alpha y_\mu \]
of bi-degree (1, 1) in the polynomial ring \( R[x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}] \), irreducible and linearly independent if \( m_2 \geq m_1 + 2 \) by Lemma 29.
Before proving the theorem, we first introduce a generalized spanning property. For \( n = 1, \ldots, m_1 \), we define the linear map
\[ S^x_n : R^{m_2} \to R^n \]
(12.1)
\[ S^x_n(y) = (p_1(x, y), \ldots, p_n(x, y)) \]
for a fixed \( x \), and the linear map
\[ S^y_n : R^{m_2} \to R^n \]
(12.2)
\[ S^y_n(x) = (p_1(x, y), \ldots, p_n(x, y)) \]
for a fixed \( y \).

**Definition 45.** We say that the \( n \)-spanning property holds if there is an \( x \in R^{m_2} \) such that \( S^x_n \) is surjective and there is a \( y \in R^{m_2} \) such that \( S^y_n \) is surjective.

Note that when \( n = m_1 \), this definition agrees with that of the spanning property in Definition 8 for the second fundamental form (see Remark 9). As for the spanning property, the \( n \)-spanning property is an open condition.

We now set up an induction procedure toward our solution to (8.1) and the spanning property.

**Induction hypothesis \( S(n) \)**

(I): \( p_1, \ldots, p_n, n \leq m_1 \), being irreducible and linearly independent imply that \( p_1^C, \ldots, p_n^C \) form a regular sequence.

(II): \( V_n := \{ z = (x, y) \in R^{m_2} \times R^{m_2} : p_a(z) = 0, a = 1, \ldots, n \} \)
and \( V_n^C := \{ z = (x, y) \in C^{m_2} \times C^{m_2} : p_a^C(z) = 0, a = 1, \ldots, n \} \)
satisfy \( \dim_R(V_n) = \dim_C(V_n^C) = 2m_2 - n \), where \( \dim_R V_n \) is the maximal dimension of all the irreducible components of \( V_n \).

(III): \( I_n := (p_1^C, \ldots, p_n^C) \) is a prime ideal.

(IV): The \( n \)-spanning property is true.

Let \( J_n \) be the subvariety of \( V_n^C \) where the Jacobian matrix of \( p_1^C, \ldots, p_n^C \) is of rank < \( n \). Proposition 43 points out that \( \text{codim}(J_n) \geq 2 \) plays a decisive role in determining the primeness of \( I_n \). We will establish in the next section the following estimate.

**Proposition 46.** Assume \( m_2 \geq m_1 + 2 \). If \( m_2 \geq 2m_1 \), then \( \text{codim}(J_n) \geq 2 \) for all \( n \leq m_1 \). If \( m_2 = 2m_1 - 1 \), then \( \text{codim}(J_n) \geq 2 \) for all \( n \leq m_1 - 1 \) whereas \( \text{codim}(J_{m_1}) \geq 1 \).
Assuming this proposition for the time being, let us prove the classification theorem of this paper.

**Theorem 47** (Classification). If \(m_2 \geq 2m_1 - 1\), then the isoparametric hypersurface is of FKM-type.

**Proof.** When \(m_1 = 1\), then \(a = 1, p = 2\) and equations (5.6) through (5.10) simplify sufficiently that one easily shows that there exists a second-order frame field for which

\[
F^\mu_{\alpha a + m_1} = \delta_{\alpha + m_2 \mu} = F^\mu_{aa} \\
F^\alpha_{pa} = 0 = F^\mu_{pa}
\]

for all \(\alpha, \mu\). The first line of these equations implies (8.1) and the spanning property. Hence, Theorem 24 implies Takagi’s result [30] that all such isoparametric hypersurfaces are of FKM-type.

Suppose \(m_2 \geq \max(m_1 + 2, 2m_1)\). Our strategy is to show that the induction procedure can be completed for \(n \leq m_1\). When \(n = m_1\) what we achieve out of the induction is that (8.1) and the spanning property hold true. It follows from Theorem 24 that the isoparametric hypersurface is of FKM-type.

\(\mathcal{S}(1)\) is true. (I) holds because \(p_1^C\) is irreducible by Lemma 29, and \(p_1^C\) cannot generate the polynomial ring since it is of degree 2. (II) is valid because \(p_1\) is bihomogeneous of bi-degree (1,1), and so one can easily solve for one variable in terms of the remaining ones regardless of whether the variables are real or complex. (III) is verified because \((p_1^C)\) is a prime ideal due to the irreducibility of \(p_1^C\). (IV) is also clear since \(p_1 \neq 0\).

Suppose \(\mathcal{S}(n - 1)\) is true for \(n - 1 \leq m_1\). We show \(\mathcal{S}(n)\) is true if \(n \leq m_1\). Now, (I) comes from Proposition 39, so that the same proposition allows us to conclude that \(\mathbb{C}[x_1, \ldots, x_{m_2}, y_1, \ldots, y_{m_2}]/(p_1^C, \ldots, p_n^C)\) is Cohen-Macaulay.

We wish to establish (II) next. To this end, note first that \(V_n^C\) is of equal dimension \(2m_2 - n\) on all irreducible components, because \(V_n^C\) is the intersection of the irreducible \(V_{n-1}^C\) and the irreducible hypersurface defined by \(p_n^C = 0\). It follows that the real variety \(V_n\) has the property

\[
\dim_{\mathbb{R}}(V_n) \leq \dim_{\mathbb{C}}(V_n^C) = 2m_2 - n,
\]

because as established in Lemma 31, \(V_n\) is a real subvariety of \(V_n^C\) and any real subvariety is of dimension at most half the (real) dimension of \(V_n^C\). We claim that there is a component of \(V_n\) having dimension \(2m_2 - n\) so that

\[
\dim_{\mathbb{R}}(V_n) = \dim_{\mathbb{C}}(V_n^C),
\]
which will establish (II). To prove the claim, consider
\[ V_n \xrightarrow{\iota} \mathbb{R}^{m_2} \times \mathbb{R}^{m_2} \xrightarrow{\pi_1} \mathbb{R}^{m_2}, \]
where \( \iota \) is the natural embedding and \( \pi_1 \) is the projection onto the first summand. Note that \((x, y) \in (\pi_1 \circ \iota)^{-1}(x)\) precisely when \( y \) belongs to the kernel of the linear map \( S_{n}^{x} \), which has dimension \( \geq m_2 - n > 0 \); in particular, \( \pi_1 \circ \iota \) is surjective. The set \( \mathcal{L} \) of \( x \) where the dimension of the kernel of \( S_{n}^{x} \) achieves the minimum value \( t \) is Zariski open. Since \( \pi_1 \circ \iota \) is surjective, one of the irreducible components \( W \) of \( V_n \) must be mapped onto an open subset of \( \mathcal{L} \) by Sard’s theorem. Around a regular value \( x \) of \( \pi_1 \circ \iota \) in \( \mathcal{L} \) we know \( V_n \) is a product with fiber \( \mathbb{R}^{t} \), which is therefore contained in the irreducible \( W \). Then since \( t \geq m_2 - n \), we have
\[ \text{dim}(W) = m_2 + t \geq m_2 + m_2 - n = 2m_2 - n. \]
Therefore
\[ \text{dim}_{\mathbb{R}}(V_n) = 2m_2 - n = \text{dim}_{\mathbb{C}}(V_n^{\mathbb{C}}), \]
which proves (II).

Now that \( \text{dim}(W) = 2m_2 - n \), the fact that \( V_n \) is a product with fiber \( \mathbb{R}^{t} \) around the regular value \( x \) gives that
\[ \text{dim}((\pi_1 \circ \iota)^{-1}(x)) = m_2 - n. \]
That is, \( S_n^{x} \) spans \( \mathbb{R}^{n} \). Likewise, there is some \( y \neq 0 \) in \( \mathbb{R}^{m_2} \) such that \( S_n^{y} \) spans \( \mathbb{R}^{n} \) if we consider the projection \( \pi_2 : \mathbb{R}^{m_2} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_2} \) onto the second summand. In conclusion, we have shown that (IV) is true.

To finish the induction, we must show that \( I_n \) is a prime ideal so that (III) holds. Proposition 43 and Proposition 46 tell us that this is true if \( V_n^{\mathbb{C}} \) is connected, which is the case because \( V_n^{\mathbb{C}} \) is a cone. In fact, if \( z \) and \( w \) are any two points in \( V_n^{\mathbb{C}} \), then the real lines from \( z \) to the origin and from the origin to \( w \) are in \( V_n^{\mathbb{C}} \), thus showing that \( V_n^{\mathbb{C}} \) is path connected.

Thus, by Propositions 43 and 46, the induction procedure is completed.

Setting \( n = m_1 \) in the induction, we obtain the spanning property in Definition 8 by induction item (IV). Note also that \( P_{b}V_{m_1} \) is \( \mathcal{D} \) defined in (10.3), and \( P_{b}V_{m_1} \) is not empty by induction item (II), which says that \( \text{dim}_{\mathbb{R}}(V_{m_1}) = 2m_2 - m_1 > m_2 \), and thus \( V_{m_1} \) contains more than \( \{0\} \times \mathbb{R}^{m_2} \cup \mathbb{R}^{m_2} \times \{0\} \).

We are only left with handling (8.1). By Proposition 28 we know \( \overline{p}_{a}, 1 \leq a \leq m_1 \), vanish on \( P_{b}V_{m_1} \) so that \( \overline{p}_{a}|_{V_{m_1}} \equiv 0 \), which warrants that \( \overline{p}_{a}|_{V_{m_1}^{\mathbb{C}}} \equiv 0 \) in view of the induction item (II) and Lemma 31, so
that \( p^a \in I_{m_1} \) by the induction item (III). Hence there are complex polynomials \( \tau_{ab} \), \( 1 \leq a, b \leq m_1 \), such that

\[
\overline{p}_a^C = \sum_{b=1}^{m_1} \tau_{ab} p_b^C.
\]

As shown in the proof of Proposition 11, we may assume that the \( \tau_{ab} \) are constant polynomials, since each of the polynomials \( \overline{p}_a^C \) and \( \overline{p}_b^C \) is of bi-degree \((1,1)\). Restricting to the real variables we obtain

\[
\overline{p}_a = \sum_{b=1}^{m_1} f_{ab} p_b
\]

for some real constants \( f_{ab} \). The above argument establishes this at every point of the open set \( U \) on which the frame is defined. By Proposition 11, after a possible change of second order frame field along \( \mathbf{x} \) on \( U \), equation (8.1) holds on \( U \). Theorem 24 then finishes the proof in the case \( m_2 \geq \max(m_1 + 2, 2m_1) \).

When \( m_2 \geq m_1 + 2 \) and \( m_2 = 2m_1 - 1 \), we can only conclude that \( V_{m_1}^C \) is a reduced variety since \( \text{codim}(J_{m_1}) \geq 1 \). Now, \( p_1^C, \ldots, p_{m_1}^C \) is still a regular sequence. From the proof of (II) above, the (real) \( V_{m_1} \) is of dimension \( 2m_2 - m_1 \). Let \( W \) be an irreducible component of \( V_{m_1}^C \) that contains an irreducible component \( \bar{V} \) of \( V_{m_1} \) of dimension \( 2m_2 - m_1 \).

By Proposition 28 all \( \overline{p}_i \) vanish on \( V \). Then all \( \overline{p}_i^C \) vanish on \( W \) by Lemma 31. Hence, we may pick a generic smooth point \( z \) of \( W \) for the Nullstellensatz to be true at \( z \). That is, \( W \) is (transversally) cut out by the ideal \( (p_1^C, \ldots, p_{m_1}^C) \) localized and still reduced at \( z \), because, in algebraic terms, localization at the maximal ideal corresponding to \( p \) in the polynomial ring preserves Cohen-Macaulayness [11, p 456]. In other words, we obtain

\[
\overline{p}_i^C q_i = \sum_{j=1}^{m_1} s_{ij} p_j^C
\]

for some local functions \( s_{ij} \) at \( z \), i.e., \( s_{ij} = r_{ij}/q_i \) with \( r_{ij} \) and \( q_i \) polynomials and \( q_i(z) \neq 0 \). Equivalently,

\[
\overline{p}_i^C q_i = \sum_{j=1}^{m_1} r_{ij} p_j^C.
\]

Let the \((x, y)\)-coordinates of \( z \) be \((h_1, \ldots, h_{m_2}, k_1, \ldots, k_{m_2})\) and set \( X_\alpha = x_\alpha - h_\alpha \) and \( Y_\mu = y_\mu - k_\mu \). Now, since \( p_a = \sum_{\alpha, \mu} F_{aa}^\mu x_\alpha y_\mu \), we substitute \((X_\alpha, Y_\mu)\) into the above Nullstellensatz equation to compare the 1st-order terms of \((X_\alpha, Y_\mu)\) to conclude \( k_\mu F_{aa}^\mu = \sum_b r_{\mu ab} F_{ab}^\mu \) and \( h_\alpha F_{aa}^\mu = \sum_b s_{\mu ab} F_{ab}^\mu \) for some constants \( r_{\mu ab} \) and \( s_{\mu ab} \). We may assume that none of the \( h_\alpha \) or \( k_\mu \) are zero by performing a generic linear
transformation. Then, one more time

\[ F_{\mu}^a = \sum_{b=1}^{m_1} f_{ab} F_{\mu}^{ab} \]

for some constants \( f_{ab} \).

When \((m_1, m_2) = (2, 3)\), Ozeki-Takeuchi [25, II] proved that \( p_1, p_2 \) are still irreducible and relatively prime, so that they form a regular sequence. Moreover, we will show in Remark 53 that \( \text{codim}(J_2) = 1 \). We are done by the preceding arguments.

**Remark 48.** In contrast, for \( m_1 = m_2 = 2 \) of non-FKM-type, we have two pairs of \((p_1, p_2)\) depending on which one of the two focal submanifolds is referred to as \( M_+ \). One pair of \((p_1, p_2) = (0, 0)\). The other pair is \((2x_2y_1 - 2x_1y_2, -2x_1y_1 - 2x_2y_2)\), out of which the real bi-projective variety \( P_2V_2^b \) is empty whereas the complex bi-projective variety \( P_2V_2^c \) consists of four points \([1 : \pm \sqrt{-1}] \times [1 : \pm \sqrt{-1}]\). This case fails to satisfy Proposition 32 miserably.

### 13. The estimate

We now prove Proposition 46 to complete the classification theorem in the preceding section. Recall for \( V_n^c \), its subvariety \( J_n \) is where the Jacobian matrix of \( p_1^c, \ldots, p_n^c \) fails to be of rank \( n \). From now on \( S_n \) and \( S_n^\mu \) in (12.1) and (12.2) will be set in the complex category.

**Lemma 49.** Notation is as in (6.5). For any choice of \( a \in \{1, \ldots, m_1\} \), there is an orthonormal basis in \( V_+ \) and an orthonormal basis in \( V_- \) such that relative to these bases we have

1. \( B_a = C_a \) with rank \( r \leq m_1 \), and
2. \( A_a = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix} \), where \( \Delta \) is an \( r \times r \) matrix in the block form \( \Delta = \text{diag}(\Delta_1, \Delta_2, \Delta_3, \ldots) \), in which \( \Delta_1 = 0 \) and \( \Delta_i, i \geq 2 \), are nonzero skew-symmetric matrices in the block form \( \Delta_i = \text{diag}(\Theta_i, \Theta_i, \ldots) \) with \( \Theta_i \) a 2-by-2 matrix of the form \( \begin{pmatrix} 0 & f_i \\ -f_i & 0 \end{pmatrix} \).

**Proof.** We know \( B_a : V_0 \rightarrow V_+ \), so that \( B_a^t B_a : V_+ \rightarrow V_+ \). Pick an orthonormal basis \( X_1, \ldots, X_{m_2-r}, Y_1, \ldots, Y_r \) of \( V_+ \) for some \( r \) such that

\[
B_a^t B_a : \begin{cases} X_t \mapsto 0, \\ Y_s \mapsto (\sigma_s)^2 Y_s, \end{cases}
\]

where \( 1 \leq t \leq m_2 - r, 1 \leq s \leq r \) and \( \sigma_s > 0 \). Now \( B_a(X_t) = 0 \) because \( \text{Ker}(B_a) \cap \text{Im}(B_a) = 0 \); hence \( X_t \in \text{Ker}(B_a) \). That is, \( \text{Ker}(B_a) \) is the
eigenspace of $B_a^t B_a$ with eigenvalue zero. On the other hand, we know $(\text{Ker}(t B_a))^\perp = \text{Im}(B_a)$. So the eigenspace decomposition of $B_a^t B_a$ is

$$V_+ = \text{Ker}(t B_a) \oplus \text{Im}(B_a)$$

with $X_1, \ldots, X_{m_2-r}$ spanning the first summand and $Y_1, \ldots, Y_r$ spanning the second. As a result, it follows that $r = \text{rank}(B_a)$. Likewise,

$$V_0 = \text{Ker}(B_a) \oplus \text{Im}(t B_a).$$

We know from above that $t B_a(X_i) = 0$ and we set

$$(13.2) \quad t B_a : Y_s \mapsto \sigma_s W_s$$

for some $W_s$. An easy calculation shows $W_i W_j = \delta_{ij}$ so that $W_1, \ldots, W_r$ form an orthonormal basis of $\text{Im}(t B_a)$. In conclusion,

$$V_0 = \text{Ker}(B_a) \oplus \text{Im}(t B_a),$$

where $W_1, \ldots, W_r$ span the second summand and we let $Z_1, \ldots, Z_{m_1-r}$ be an orthonormal basis generating the first. We find by (13.1) that

$$(13.3) \quad B_a : Z_t \mapsto 0,$$

$$\quad : W_s \mapsto \sigma_s Y_s.$$ 

We calculate to see that $t B_a B_a : V_0 \mapsto V_0$ satisfies

$$(13.4) \quad t B_a B_a : Z_t \mapsto 0,$$

$$\quad : W_s \mapsto (\sigma_s)^2 W_s.$$ 

Now consider

$$C_a : V_0 \mapsto V_-.$$ 

In the same manner as above for $B_a$, we get $V_0 = \text{Ker}(C_a) \oplus \text{Im}(\text{t} C_a)$ with

$$(13.5) \quad C_a : Z_t^* \mapsto 0,$$

$$\quad : W_s^* \mapsto \sigma_s^* Y_s^*,$$

where $Z_1^*, \ldots, Z_{m_1-p}^*$ span $\text{Ker}(C_a)$ and $W_1^*, \ldots, W_p^*$ span $\text{Im}(C_a)$ for some $p$. However,

$$t C_a C_a = t B_a B_a$$

by the first equation of (5.6), we thus obtain $\text{Ker}(B_a) = \text{Ker}(C_a)$ and $\text{Im}(t B_a) = \text{Im}(t C_a)$. In particular, $p = r$ and we may take $Z_1, \ldots, Z_{m_1-r}$ to be identical with $Z_1^*, \ldots, Z_{m_1-r}^*$, and $W_1, \ldots, W_r$ to be identical with $W_1^*, \ldots, W_r^*$. Therefore (13.3) and (13.5) imply that we can pick a basis of $V_+$ and a basis of $V_-$ relative to which the matrices of these operators, denoted by the same letters as the operators, satisfy

$$(13.6) \quad B_a = C_a.$$
because from
\[
^tC_a C_a : Z_t^* \longrightarrow 0,
\]
\[
: W_s^* \longrightarrow (\sigma_s^*)^2 W_{\bar{s}}^*
\]
and \( W_s = W_{\bar{s}}^* \), we know \( (\sigma_s)^2 = (\sigma_{\bar{s}})^2 \), and hence we may assume \( \sigma_s = \sigma_{\bar{s}} \) by adjusting the basis in \( V_- \).

The second and the fourth equations of (5.6) together with (13.6) yield

\[
A_a^t A_a = A_a A_a^t = I - 2B_a^t B_a.
\]

We have three more equations

\[
B_a^t B_a A_a + A_a B_a^t B_a = 0,
\]
\[
B_a^t B_a A_a + A_a B_a^t B_a = 0,
\]
\[
B_a^t A_a B_a + B_a A_a B_a = 0,
\]
which can be derived from (13.6) and the three diagonal blocks of (6.7).

Let
\[
A_a = \begin{pmatrix} \alpha & \beta \\ \gamma & \mu \end{pmatrix}
\]

where \( \alpha \) is of size \((m_2 - r) \times (m_2 - r)\) and \( \mu \) is of size \( r \times r \). Let \( \sigma = \text{diag}(\sigma_1, \ldots, \sigma_r) \) be the diagonal matrix with the indicated diagonal entries so that by (13.2) and (13.3), \( B_a \) and \( B_a^t \) are of the same form

\[
(13.11) \quad \begin{pmatrix} 0 & 0 \\ 0 & \sigma \end{pmatrix}
\]

with \( B_a^t B_a = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^2 \end{pmatrix} \) of the same block sizes as \( A_a \). From (13.8) we obtain

\[
(13.12) \quad \beta = \gamma = 0,
\]
\[
(13.13) \quad \sigma^2 (t^t \mu) = -\mu \sigma^2.
\]

Moreover from (13.7) we see

\[
(13.14) \quad \alpha^t \alpha = I,
\]
\[
(13.15) \quad \mu^t \mu = (t^t \mu \mu) = I - 2\sigma^2.
\]

Similarly, (13.9) yields

\[
(13.16) \quad \sigma^2 \mu = -t^t \mu \sigma^2,
\]
and (13.10) gives

\[
(13.17) \quad \sigma^t \mu \sigma = -\sigma \mu \sigma.
\]
With (13.13) and (13.16) we deduce
\[ \mu_{ij} = -(\sigma_i/\sigma_j)^2 \mu_{ji}, \]
and
\[ \mu_{ji} = -(\sigma_i/\sigma_j)^2 \mu_{ij}. \]
We therefore conclude
\[ \mu_{ij} = 0 \text{ if } \sigma_i \neq \sigma_j, \]
and
\[ \mu_{ij} = -\mu_{ji} \text{ if } \sigma_i = \sigma_j. \]
In other words,
\[ A_a = \begin{pmatrix} \alpha & 0 \\ 0 & \mu \end{pmatrix} \]
with \( \alpha \mu = I \) and \( \mu \) is in blocked form
\[ \mu = \text{diag}(\Delta_1, \Delta_2, \Delta_3, \cdots), \]
where all the \( \Delta_i \) are skew-symmetric such that the number of \( \Delta_i \) is the number of different non-zero eigenvalues of \( B_a^i B_a \). Then (13.17) is automatically satisfied. Now by the skew-symmetry of \( \mu \) and (13.15) we derive
\[ \Delta_i^2 = -(1 - 2\sigma_i^2)I. \]
In view of (13.14) and the skew-symmetry of \( \mu \) we can perform an orthonormal basis change so that \( \alpha = I \) and
\[ \Delta_i = \text{diag}\left( \begin{pmatrix} 0 & r_1 \\ -r_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & r_2 \\ -r_2 & 0 \end{pmatrix}, \cdots \right). \]
Thus (13.18) implies \( r_1^2 = r_2^2 = \cdots = 1 - 2\sigma_i^2 \), and so
\[ \Delta_i = \sqrt{1 - 2\sigma_i^2} \text{ diag}\left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots \right) \]
if \( 1 - 2\sigma_i^2 > 0 \). We set \( \Delta_1 \equiv 0 \) so that \( \sigma_1 = 1/\sqrt{2} \). We are done.

**Corollary 50.** \( \dim(\text{Ker}(A_a)) = \dim(\Delta_1) \leq r = \text{rank}(B_a) \leq m_1. \)

**Remark 51.** When \( (m_1, m_2) = (2, m_2), m_2 \geq 3 \), Ozeki and Takeuchi showed [25, II, p.49], that \( r \) given in Lemma 49 is 1, essentially by exploring the fact that \( p_1 \) and \( p_2 \) form a regular sequence in the spirit of Example 35 above. It follows immediately from Lemma 49 that we have \( \Delta = 0 \) and so \( A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \) as given in [25, II, p. 51]. With this
it is not hard to see that $A_2 = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ of the same block sizes as $A_1$ with $B = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where $I$ in $B$ is of size $l \times l$ and $m_2 = 2l + 1$.

**Proof of Proposition 46.** We must estimate the codimension in $V_n^C$ of $J_n = \{(x, y) \in V_n^C : dp_1^C \wedge \cdots \wedge dp_n^C = 0\}$.

We first estimate the dimension of the subvariety $Z_n$ of $C^{m_2} \times C^{m_2}$, defined to be the locus of points where the Jacobian matrix of $p_1^C, \ldots, p_n^C$ is of rank $< n$. At $(x, y) \in Z_n$, the differentials $dp_1^C, \ldots, dp_n^C$ are linearly dependent, i.e., there exists $[c_1 : \cdots : c_n] \in CP^{n-1}$, depending on $(x, y)$, such that

$$0 = \sum_{a=1}^{n} c_a dp_a^C = \sum_{a} \sum_{a, \mu} c_a F_{a \mu}^\alpha y_{\mu} dx_\alpha + \sum_{\mu} \sum_{a, \alpha} c_a F_{a \alpha}^\mu x_{\alpha} dy_{\mu},$$

which requires that the coefficients of $dx_\alpha$ be zero and the coefficients of $dy_{\mu}$ be zero. Thus

$$Z_n = \{(x, y) \in C^{m_2} \times C^{m_2} : \exists [c_1 : \cdots : c_n], \sum_a c_a^t A_a x = \sum_a c_a A_a y = 0\}.$$

In order to estimate dim$Z_n$, let us define, for a fixed $[c_1 : \cdots : c_n] \in CP^{n-1}$,

$$Z_{(c_1, \ldots, c_n)} := \{(x, y) \in C^{m_2} \times C^{m_2} : \sum_a c_a^t A_a x = \sum_a c_a A_a y = 0\}.$$

Consider the incidence space $Y_n$ in $CP^{n-1} \times C^{m_2} \times C^{m_2}$ given by

$$(13.19) \quad Y_n = \{([c_1 : \cdots : c_n], x, y) : (x, y) \in Z_{(c_1, \ldots, c_n)}\}.$$

The standard projection of $Y_n$ to $C^{m_2} \times C^{m_2}$ maps $Y_n$ onto $Z_n$. Let $\pi$ be the standard projection of $Y_n$ to $CP^{n-1}$. Then with respect to $\pi$ we have

$$(13.20) \quad \dim(Z_n) \leq \dim(Y_n) \leq \dim(\text{base}) + \dim(\text{fiber}),$$

where $\dim(\text{fiber})$ is the maximal dimension of all fibers. We first estimate the dimension of the fibers $\pi^{-1}\{[c_1 : \cdots : c_n]\} = Z_{(c_1, \ldots, c_n)}$. In fact, it comes down to estimating the dimension of

$$T_{(c_1, \ldots, c_n)} := \{y \in C^{m_2} : \sum_a c_a A_a y = 0\}$$

for a fixed $[c_1 : \cdots : c_n]$, because

$$(13.21) \quad \dim(\ker(\sum_a c_a^t A_a)) = \dim(\ker(\sum_a c_a A_a)),$$
thus giving us the estimate

$$\dim(Z(c_1, \ldots, c_n)) \leq 2 \dim(T(c_1, \ldots, c_n)).$$

**Remark 52.** Let us examine the case \((m_1, m_2) = (2, m_2), m_2 \geq 3\), before we proceed. By the above standard matrix form of \(A_1\) and of \(A_2\) in Remark 51 we see that for \(t = (t', s) \in \mathbb{C}^{m_2}\), where \(s \in \mathbb{C}\),

\[
A_1 \quad t' = t'(z, 0), \quad A_2 \quad t'(z, s) = t'(Bz, 0).
\]

Hence \(\sum_{a=1}^{n} c_a A_a y = 0\) precisely when \(z = 0\) or \(z\) is an eigenvector of \(B\), with eigenvalue \(\epsilon \sqrt{-1}\), where \(\epsilon = \pm\). In other words, when \([c_1 : c_2] = [\epsilon \sqrt{-1} : 1]\) in \(\mathbb{C}P^1\), then

\[
Z(c_1, c_2) = \{(u, -\epsilon \sqrt{-1} u, t), (v, \epsilon \sqrt{-1} v, s)) : u, v \in \mathbb{C}^l, s, t \in \mathbb{C}\},
\]

and \(Z(c_1, c_2) = \{(0, 0, t), (0, 0, s)) : s, t \in \mathbb{C}\} for other values of \([c_1 : c_2]\).

Thus

\[
Z_2 = Z(\sqrt{-1}, 1) \cup Z(-\sqrt{-1}, 1),
\]

and so

\[
\dim(Z_2) = 2l + 2 = m_2 + 1.
\]

We continue on now to estimate the dimension of \(Z(c_1, \ldots, c_n)\).

**Case (1).** \(c_1, \ldots, c_n\) are either all real or all purely imaginary. Say it is the latter, so that \(c_k = \sqrt{-1} d_k\) with \(d_k\) real. Then for \(y \in T(c_1, \ldots, c_n)\), we have

\[
\sum_{k=1}^{n} d_k A_k y = 0.
\]

However, the second fundamental form \(S\) has the property

\[
d_1 S_{e_1} + \cdots + d_n S_{e_n} = \sqrt{d_1^2 + \cdots + d_n^2} e,
\]

where

\[
e = (d_1 e_1 + \cdots + d_n e_n)/\sqrt{d_1^2 + \cdots + d_n^2}.
\]

We may therefore rename \(e\) to be \(e_1\) in the normal basis, and so by restricting to the \(A\)-block in the matrix of \(S\) we see that \(S e y = 0\) comes down to, after the renaming, \(A_1 y = 0\). Corollary 50 then establishes that

\[
\dim(T(c_1, \ldots, c_n)) \leq r \leq m_1
\]

and

\[
\dim(Z(c_1, \ldots, c_n)) \leq 2 \dim T(c_1, \ldots, c_n) \leq 2m_1
\]
Case (2). \( c_1, \ldots, c_n \) are not all real and not all purely imaginary. Write
\[
c_k = \alpha_k + \sqrt{-1} \beta_k,
\]
where not all \( \alpha_k \) and not all \( \beta_k \) are zero. Then
\[
c_1 S_{e_1} + \cdots + c_n S_{e_n} = (\alpha_1 S_{e_1} + \cdots + \alpha_n S_{e_n}) + \sqrt{-1}(\beta_1 S_{e_1} + \cdots + \beta_n S_{e_n}).
\]
As in Case (1), we know \( \alpha_1 S_{e_1} + \cdots + \alpha_n S_{e_n} \) is a multiple of \( S_e \) for some unit vector \( e \). Hence without loss of generality we may assume, after renaming \( e \) to be \( e_1 \), that
\[
c_1 S_{e_1} + \cdots + c_n S_{e_n} = \alpha_1 S_{e_1} + \sqrt{-1}(\beta_1 S_{e_1} + \cdots + \beta_n S_{e_n}).
\]
On the other hand \( \beta_2 S_{e_2} + \cdots + \beta_n S_{e_n} \) is a multiple of \( S_f \) for some unit vector \( f \) perpendicular to \( e_1 \). We rename \( f \) to be \( e_2 \) so that we may assume without loss of generality that
\[
c_1 S_{e_1} + \cdots + c_n S_{e_n} = (\alpha_1 + \sqrt{-1} \beta_1) S_{e_1} + \sqrt{-1} \beta_2 S_{e_2}.
\]
By restricting to the \( A \)-block in \( S \) again we see that \( (\sum_a c_a A_a) y = 0 \) is reduced to
\[
\beta_2 A_2 y = \sqrt{-1}(\alpha_1 + \sqrt{-1} \beta_1) A_1 y.
\]
We may assume both coefficients are nonzero, or else we would be back to Case (1). Hence we are now handling
\[
(A_2 - z A_1) y = 0
\]
for some nonzero \( z \in \mathbb{C} \). By Lemma 49, we may assume \( A_1 = \begin{pmatrix} I & 0 \\ 0 & \Delta \end{pmatrix} \).
Write
\[
A_2 = \begin{pmatrix} \Theta & A \\ \Omega & \Gamma \end{pmatrix}
\]
of the same block sizes as \( A_1 \). By the second equation of (5.6), which is
\[
A_2^t A_1 + A_1^t A_2 + 2(B_2^t B_1 + B_1^t B_2) = 0,
\]
we obtain
\[
\Theta + \Theta^t = 0
\]
when we invoke (13.11). If we write
\[
y = \Theta^t (u, v), \quad u \in \mathbb{C}^{m_2-r}, \quad v \in \mathbb{C}^r
\]
then part of (13.25) reads,
\[
(z I - \Theta) u = \Lambda v.
\]
Consider the map \( G : \mathbb{C}^{m_2} \rightarrow \mathbb{C}^{m_2-r} \) given by
\[
G : (u, v) \mapsto (z I - \Theta) u - \Lambda v.
\]
The kernel of $G$ consists of all $y = \ell(u,v)$ satisfying (13.27). If $z$ is not an eigenvalue of $\Theta$, then the rank of $G$ is at least the rank of $zI - \Theta$, which is $m_2 - r$. Thus, the rank of $G$ is $m_2 - r$, so that the kernel of $G$ has dimension $r$. On the other hand if $z$ is an eigenvalue of $\Theta$, then because $\Theta$ is skew-symmetric by (13.26), the rank of $zI$ is at least $(m_2 - r)/2$ due to the fact that a nonzero eigenvalue of $\Theta$ is purely imaginary, and its conjugate is also an eigenvalue of $\Theta$. It follows that the rank of $G$ is no less than $(m_2 - r)/2$, so that its kernel is of dimension $\leq (m_2 + r)/2$. The upshot is that, since $r \leq m_1$ and since $\dim(T_{c_1, \ldots, c_n})$ is an integer, we have arrived at the estimate
\[
\dim(T_{(c_1, \ldots, c_n)}) \leq [(m_2 + r)/2] \leq [(m_2 + m_1)/2] = (m_2 + m_1 - 1)/2,
\]
where $[p]$ is the greatest integer in the number $p$, and the last equality is true because $m_2 + m_1$ is an odd number when $2 \leq m_1 < m_2$ by a result of Münzner [22, II]. Therefore,
\[
(13.28) \quad \dim(\text{fiber}) = \dim(Z_{(c_1, \ldots, c_n)}) \leq 2 \dim(T_{(c_1, \ldots, c_n)}) \leq m_2 + m_1 - 1.
\]
This estimate is sharp in light of (13.24). Note that $m_2 + m_1 - 1$ is greater than the upper bound $2m_1$ for $\dim(Z_{(c_1, \ldots, c_n)})$ in Case (1), since $m_2 \geq m_1 + 2$, by assumption.

We next stratify the incidence space $Y_n$ of (13.19) in another way as follows. We let $s \leq m_2$ be the largest integer for which $\sum_{i=1}^n c_i A_i$ is of rank $s$ for some, and hence for generic, $[c_1 : \cdots : c_n]$, the set of which constitute a Zariski open set $U$ of $\mathbb{C}P^{n-1}$. A look at Corollary 50 shows that $s \geq m_2 - m_1$, so that for $[c_1 : \cdots : c_n]$ in $U$,
\[
\rank(\sum_{i=1}^n c_i A_i) = s \geq m_2 - m_1,
\]
and thus, by (13.21),
\[
\dim(\text{fiber}) = \dim(Z_{(c_1, \ldots, c_n)}) = \dim(\ker(\sum_{i=1}^n c_i A_i)) + \dim(\ker(\sum_{i=1}^n c_i^t A_i))
\]
\[
= 2(m_2 - \rank(\sum_{i=1}^n c_i A_i)) = 2(m_2 - s) \leq 2m_1.
\]
It follows that over $U$, (13.20) extends to
\[
(13.29) \quad \dim(\text{fiber}) + \dim(\text{base}) \leq 2m_1 + (n - 1).
\]
On the other hand, over a subvariety $W$, contained in $\mathbb{C}P^{n-1}$, of dimension $\leq n-2$, the rank of $\sum_{i=1}^n c_i A_i$ is less than $s$. In view of (13.28),
we have that over $W$

\begin{equation}
\dim(\text{fiber}) + \dim(\text{base}) \leq \dim(\text{fiber}) + n - 2
\leq m_1 + m_2 - 1 + n - 2 = m_1 + m_2 + n - 3.
\end{equation}

(13.30)

The part of $Y_n$ over $U$, call it $A$, is irreducible because each fiber over $U$ is a Euclidean space of a fixed dimension, whereas the part over $W$, call it $B$, is Zariski closed in $Y_n$. It follows that the closure of $A$, call it $\overline{A}$, in $Y_n$ is an irreducible component of $Y_n$, and the closure of $B$ not in $\overline{A}$ constitutes the remaining irreducible components in $Y_n$. Therefore, the larger of the two upper bounds in (13.29) and (13.30) will be an upper bound for the dimension of $Y_n$. However, $2m_1 + n - 1 \leq m_1 + m_2 + n - 3$, because $m_2 \geq m_1 + 2$. We conclude that over $\mathbb{CP}^{n-1}$

\begin{equation}
\dim(Y_n) \leq m_1 + m_2 + n - 3
\end{equation}

if $m_2 \geq m_1 + 2$.

Now, Lemma 29 says that $p_1^C, \ldots, p_n^C$ are linearly independent. Consider the map

$$f : (x, y) \in \mathbb{C}^{m_2} \times \mathbb{C}^{m_2} \mapsto (p_1^C(x, y), \ldots, p_n^C(x, y)) \in \mathbb{C}^n.$$ 

Note that $Z_n$ is the singular point set of $f$ and $J_n = f^{-1}(0) \cap Z_n$.

Let us make a general remark about a refined version of Sard’s theorem before proceeding. In the following, the irreducible objects $X$ in the projectivized domain of $f$, which is $\mathbb{CP}^{m_2-1} \times \mathbb{CP}^{m_2-1}$ with $PV_n^C$ removed, are all quasi-projective, i.e., are all Zariski open subsets of projective varieties. $f|_X$ can be considered as a rational map into $\mathbb{CP}^{n-1}$. So, by [21, p50], there is a Zariski open set $\mathcal{O}$ of $f(X)$ in $\mathbb{C}^n$ (with the origin excluded) such that $\dim(f|_X^{-1}(y)) = \dim(X) - \dim(f(X))$ is a constant for all $y \in \mathcal{O}$; we call it the generic fiber dimension of $f|_X$. Furthermore, $\text{codim}(f(X) \setminus \mathcal{O}) \geq 2$ in $f(X)$.

Recall from (13.19) that the projection $\Pi : Y_n \rightarrow \mathbb{C}^{m_2} \times \mathbb{C}^{m_2}$ is $Z_n$. Observe that at $(x, y)$, the dimension of the kernel of the Jacobian matrix of $p_1^C, \ldots, p_n^C$ at $(x, y)$ is 1 more than the dimension of the projective space $\Pi^{-1}((x, y))$. $\mathbb{C}^{m_2} \times \mathbb{C}^{m_2}$ is stratified into locally closed sets (i.e., Zariski open sets in their respective closures) $X_{-1}, X_0, X_1, \ldots, X_{n-1}$ such that $df$ has rank $n - j - 1$ on $X_j$ ($X_j$ may be empty). Note that $\Pi$ has fiber dimension $j$ over $X_j$. Let $k$ be the first $j \geq 0$ for $X_j$ to be nonempty. Then $Z_n = \bigcup_{j=0}^{n-1} X_j$ and $\bigcup_{j=k+1}^{n-1} X_j$ is a Zariski closed set of $Z_n$. Let $U_0$ be an irreducible component of $Z_n$. The smooth part of $U_0$ consists of irreducible components of $X_k$. Then the generic rank of $f|_{U_0}$ is $n - k - 1$, so that $\dim(f(U_0)) = n - k - 1$. Set $S := U_0 \cap (\bigcup_{j=k+1}^{n-1} X_j)$. $S$ is Zariski closed of codimension at least 1 in $U_0$. The generic fiber $\mathcal{F}_0$ of $\Pi$ over $U_0$ has dimension $k$. 

We now use an inductive procedure. Suppose $U_i$ of codimension $i$ in $U_0$ has been defined and the generic hyperplanes $L_i$ chosen ($L_0$ is the empty set), in such a way that $S_i := U_i \cap S$ is of codimension at least 1 in $U_i$, so that the generic fiber dimension of $\Pi$ over $U_i$ is $k$. Let $W$ be an irreducible component of $U_i$. Observe that since $\dim(W) = \dim(U_i)$ so that $W \cap S_i$ is of codimension at least 1 in $W$, we have that $f(W \cap S_i)$ is of codimension at least 1 in $f(W)$; or else the generic fiber dimension of $f|_W$ over $f(W)$ would be reduced to a smaller number.

Now, if $f(W) \neq \{0\}$, we pick a generic hyperplane $L_{i+1}$, transversal to $L_j$, $0 \leq j \leq i$, through the origin and transversal to $f(W)$ and $f(W \cap S_i)$. (This is possible. Since $f$ and the hyperplanes $L_1, L_2, \ldots, L_i$ are all homogeneous, we may consider the cuts to be done in the projective setting, and thereby get that $L_{i+1}$ is a hyperplane through the origin.) This warrants that the cone $L_{i+1}(p_1, \ldots, p_n) = 0$ intersects $W$ and $W \cap S_i$ transversally to cut out $Q_W$ of codimension 1 in $W$ and $Q_W \cap S_i$ of codimension at least 1 in $Q_W$, as we go through each of the irreducible components $W$ with $f(W) \neq 0$. Let $U_{i+1}$ be the union of all such $Q_W$. $U_{i+1}$ is of codimension $i + 1$ in $U_0$. Furthermore, $S_{i+1} = U_{i+1} \cap S = \cup_W (Q_W \cap S_i)$ is of codimension at least 1 in $U_{i+1}$. Therefore, the generic fiber $F_{i+1}$ of $\Pi$ over $U_{i+1}$ has dimension $k = \dim(F_0)$.

On the other hand, if an irreducible component $L$ of $U_i$ satisfies $f(L) = 0$, then $\text{codim}(L) = i$ in $U_0$. We claim the generic fiber of $\Pi$ over $L$ is of dimension $n - i - 1$. This is because when $i = 0$, $f(U_0) = 0$ implies $df$ has rank 0, so that $\Pi$ has generic fiber dimension $n - 1$ over $U_0$; $k = n - 1$ in this case. If $f(U_0) \neq 0$, we go to $U_1$. If $f(L) = 0$ for some irreducible component of $U_1$, then since $df = 0$ on $L$ at its generic points, which are also generic in $U_0$, and since $df \neq 0$ on $U_0$ generically, we see $df$ has rank 1 at a generic point of $L$. That is, $\Pi$ has generic fiber dimension $n - 2$ over $L$; $k = n - 2$ in this case. Then we move to $U_2$, etc. Accordingly, we set $T_i$ to be the union of all such $L$; we have $k = n - i - 1$ for a nonempty $T_i$. In particular, $T_i$ are all empty for $0 \leq i < n - k - 1$. The first possibly nontrivial one is thus $T_{n-k-1}$.

Continuing in this fashion, the next-to-last $f(U_{n-k-2})$ consists of finitely many lines through the origin. Then the last cut by the generic hyperplane $L_{n-k-1}$ picks up the origin of $\mathbb{C}^n$. But then $f(U_{n-k-1}) = 0$ means $T_{n-k-1} = U_{n-k-1}$. The cutting procedure ends.

Consequently, with $\dim(F_{n-k-1}) = k$ and $\text{codim}(U_{n-k-1}) = n - k - 1$ in $U_0$, we have

$$\dim(U_{n-k-1}) \leq \dim(Z_n) - (n - k - 1)$$

$$\leq \dim(Y_n) - \dim(F_{n-k-1}) - (n - k - 1)$$

$$\leq m_1 + m_2 - 2$$
by (13.31). Now, since the variety $J_n$ is the union of all $U_{n-k-1}$ as $U_0$ goes through all the irreducible components of $Z_n$, we deduce $\dim(J_n) \leq m_1 + m_2 - 2$. Hence, if $m_2 \geq m_1 + n$ (respectively, $m_2 \geq m_1 + n - 1$), then

$$\dim(J_n) \leq m_1 + m_2 - 2 \leq 2m_2 - n - 2 \leq \dim(V_n^C) - 2$$

(respectively, $\leq \dim(V_n^C) - 1$). So, if $m_2 \geq 2m_1$, then $J_n$ is of codimension at least 2 for all $n \leq m_1$. Further, if $m_2 = 2m_1 - 1$, then $J_n$ is of codimension at least 2 for all $n \leq m_1 - 1$, and $J_{m_1}$ is of codimension at least 1. This implies the statements of Proposition 46.

The classification result Theorem 47 is therefore established.

**Remark 53.** The standard matrix form of $A_1$ and of $A_2$ in the case $(m_1, m_2) = (2, m_2), m_2 \geq 3$, given in Remark 51 leads to

$$p_1 = 2 \sum_{j=1}^{l} (x_jy_j + x_{l+j}y_{l+j}), \quad p_2 = -2 \sum_{j=1}^{l} (x_jy_{l+j} - x_{l+j}y_j),$$

where $m_2 = 2l + 1$. Then $J_2 = V_2^C \cap Z_2$, which by (13.22) and (13.23) is

$$\begin{align*}
\text{(13.32)} \quad J_2 &= \{(u, -\sqrt{-1}u, t), (v, \sqrt{-1}v, s) \in Z_2 : \sum_{j=1}^{l} u_jv_j = 0\}
\end{align*}$$

where $u, v \in C^l$, $t, s \in C$, and $\epsilon = \pm$. It follows that $\dim(J_2) = \dim(Z_2) - 1 = m_2$, by (13.24). Thus, $\text{codim}(J_2) \geq 2$ in $V_2^C$ (which is of dimension $2m_2 - 2$), provided $m_2 \geq 4$.

For $m_2 = 3$, $J_2$ has codimension 1 in $V_2^C$. Indeed, in this case the bi-projective variety $P_0V_2^C$ defined by $p_1 = p_2 = 0$ in $CP^2 \times CP^2$ is made up of four irreducible components,

$$P_0V_2^C = CP_1^1 \times CP_1^1 \cup CP_1^1 \times CP_1^1 \cup \{[0 : 0 : 1]\} \times CP^2 \cup CP^2 \times \{[0 : 0 : 1]\}$$

where $CP_1^1 \hookrightarrow CP^2$ by $[u : s] \mapsto [u : \epsilon\sqrt{-1}u : s]$. Hence $(p_1, p_2)$ is not a prime ideal in $C[x_1, x_2, y_1, y_2]$ and Proposition 43 says then that codim $(J_2) \leq 1$ in $V_2^C$. In fact, in this case codim $(J_2) = 1$ in $V_2^C$, and $x_2(y_1^2 + y_2^2) \in (p_1, p_2)$, but neither $x_2$ nor $y_1^2 + y_2^2$ is in the ideal.

In view of the known classification of Takagi [30] for $m_1 = 1$, Ozeki-Takeuchi [25, II] for $m_1 = 2$, and Stolz’s result [29] on the multiplicities $m_1 \leq m_2$ that states that $(m_1, m_2) \neq (2, 2)$ or $(4, 5)$ must be that of an isoparametric hypersurface of FKM-type, we obtain from Theorem 47 that all isoparametric hypersurfaces with four principal curvatures in
spheres, whose multiplicities are not (2, 2) or (4, 5), are of FKM-type, except possibly for those whose multiplicities are one of the following 3 pairs (3, 4), (6, 9), (7, 8). The (4, 5) case also remains unclassified.

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Department of Mathematics and Computer Science, College of the Holy Cross, Worcester, Massachusetts 01610-2395
E-mail address: cecil@mathcs.holycross.edu

Department of Mathematics, Campus Box 1146, Washington University, St. Louis, Missouri 63130
E-mail address: chi@math.wustl.edu

Department of Mathematics, Campus Box 1146, Washington University, St. Louis, Missouri 63130
E-mail address: gary@math.wustl.edu