Final Examination
Math 441, Fall 2003

There is a total of 100 points.

Define the following (7 points each):

1. A Riemannian metric on a differentiable manifold M^n.

2. The Frobenius condition for a smooth k-plane distribution $\mathcal{D}$ on a smooth manifold M^n.

3. A Lie group G.

Prove the following form of the Gauss-Bonnet Theorem (30 points).

4. Let M^2 be an oriented Riemannian surface with Gaussian curvature K and area element dA. If T is a geodesic triangle, then

$$\int_T K\, dA = \sum_{i=1}^{3} \alpha_i - \pi$$

where α_i is the interior angle at the i^{th} vertex of T. You may assume that your version of Stokes's Theorem applies to T and that T lies in an open set U on which there is an oriented orthonormal moving frame e_1, e_2.

Do the following problems (worth 7 points each).

5. Let α be a closed differentiable k-form and let β be an exact differentiable r-form, on a differentiable manifold M^n. Prove that $\alpha \wedge \beta$ is exact.

6. Prove that a closed 1-form α on the sphere $S^n \subset \mathbf{R}^{n+1}$ is exact, if $n \geq 2$.

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Let $\gamma : J \to \mathbf{R}^3$ be a smooth embedded curve in $\mathbf{R}^3$ parametrized by arclength parameter $s \in J$, where J is an interval containing 0. Assume that the curvature $\kappa(s)$ is positive for every point $s \in J$. Let $T = \dot{\gamma}$, N, B be its Frenet frame, with Frenet-Serret equations

$$\dot{T} = \kappa N, \quad \dot{N} = -\kappa T + \tau B, \quad \dot{B} = -\tau N$$

where the function $\tau(s)$ is the torsion.

Let r be a positive constant such that $r < 1/\kappa(s)$ for all $s \in J$. The *tube* around γ of radius r is the map

$$\begin{aligned} &\mathbf{x} : J \times \mathbf{R} \to \mathbf{R}^3 \\ &\quad \mathbf{x}(s,t) = \gamma(s) + r(\cos t\, N(s) + \sin t\, B(s)) \end{aligned}$$

7. Prove that $\mathbf{x}$ is an immersion on $M = J \times \mathbf{R}$.

8. Prove that $e_3(s,t) = \cos t\, N(s) + \sin t\, B(s)$ is a Gauss map of $\mathbf{x}$.

9. For the orthonormal moving frame

$$e_1(s,t) = T(s), \quad e_2(s,t) = \sin t\, N(s) - \cos t\, B(s), \quad e_3(s,t)$$

find the dual coframe field θ^1, θ^2 and its Levi-Civita connection form $\omega_2^1 = -\omega_1^2$.

10. Find the second fundamental form

$$II = h_{11}\theta^1\theta^1 + 2h_{12}\theta^1\theta^2 + h_{22}\theta^2\theta^2$$

of $\mathbf{x}$ for the Gauss map e_3.

11. If $L > 0$ and $L \in J$, find the area of $\mathbf{x}(D)$, where $D = [0, L] \times [0, 2\pi] \subset M$.