Final Examination
Math 441, Fall 2004

You may use theorems from advanced calculus, such as the inverse and implicit function theorems and the rank theorem, by citation without proof.

1). Define the real projective space $P^n(\mathbf{R})$ (as a topological manifold) and prove that it is compact.

2). Prove that if $f_1, \ldots, f_n$ are C^∞ functions on a C^∞ manifold M^n such that

$$df_1(p) \wedge \cdots \wedge df_n(p) \neq 0$$

at some point $p \in M$, then there exists a chart U, x about p for which $x^i = f_i$ restricted to U. Hint: consider $F = (f_1, \ldots, f_n) : M \to \mathbf{R}^n$.

3). Let $\theta : W \subset \mathbf{R} \times S^2 \to S^2$ be the flow of the C^∞ vector field $X(x, y, z) = (0, z, -y)$ on the unit sphere S^2.

i). Find W.

ii). Find $\theta(t, (1, 0, 0))$, for every $t \in \mathbf{R}$ at which it is defined.

4). On $\mathbf{R}^3$ consider the 2-plane distribution $\mathcal{D}^\perp = \{\alpha\}$, where

$$\alpha = dx + zdy - xdz$$

and where x, y, z are the standard coordinates on $\mathbf{R}^3$. Verify whether this distribution is involutive.

5). Let U_1 and U_2 be open subsets of the C^∞ manifold M^n. Let k be an integer between 0 and n. The Mayer-Vietoris sequence is based on the short exact sequence of cochains

$$0 \to A^k(U_1 \cup U_2) \xrightarrow{i} A^k(U_1) \oplus A^k(U_2) \xrightarrow{j} A^k(U_1 \cap U_2) \to 0$$

Explain how j is defined and use a partition of unity to prove that it is onto.

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6). Let $F : G \to H$ be a Lie group homomorphism between the Lie groups G and H. Let X be a left-invariant vector field on G and let Y be the left-invariant vector field on H determined by the vector $dF_{(e)}X_{(e)} \in T_eH$. Prove that Y is F-related to X.

7). Let S^2 be the unit sphere in $\mathbf{R}^3$, let $j : S^2 \hookrightarrow \mathbf{R}^3$ be the inclusion map, and let $M^2 = \{p \in S^2 : z(p) \geq 0\}$ be the closed northern hemisphere. Let S^2 have the orientation defined by the nowhere vanishing 2-form $j^*(\iota_{(x,y,z)} dx \wedge dy \wedge dz)$, where x, y, z are the standard coordinates on $\mathbf{R}^3$.

i). State Stokes's Theorem for the 1-form $\alpha = j^*(xdy - ydz)$ on M. Explain the correct orientation on the boundary of M.

ii). Calculate $\int_{\partial M} \alpha$, where ∂M has the induced orientation.