

Geometry Qualifying Examination
 Math 441 Part Solutions
 Spring 2005

This section has 10 problem parts, each of equal value.

1. On a C^∞ manifold M^m , explain the definition of tangent vector at a point $p \in M$ as:

(a) An equivalence class of curves. A curve through $p \in M$ is a smooth map $\sigma: J \rightarrow M$, where J is an open interval about 0 in \mathbb{R} , such that $\sigma(0) = p$. Two such curves $\sigma: J \rightarrow M$ and $\tilde{\sigma}: \tilde{J} \rightarrow M$ are equivalent if $(f \circ \sigma)'(0) = (f \circ \tilde{\sigma})'(0)$, for every smooth function f defined on a nbhd of p . A tangent vector at p is an equivalence class of such curves, denoted $[\sigma]$.

(b) A derivation on some algebra. Let $C^\infty(p)$ be the algebra over \mathbb{R} of germs of C^∞ functions at p on M . Denote the germ of f at p by $[f]$. A tangent vector at p is a derivation $v: C^\infty(p) \rightarrow \mathbb{R}$; that is, an \mathbb{R} -linear map satisfying the product rule

$$v([f][g]) = f(p)v[g] + g(p)v[f]$$

- (c) In each case, explain how the derivative of a C^∞ map $F: M \rightarrow N$ between manifolds maps the tangent vectors at $p \in M$ to tangent vectors at $F(p) \in N$. $dF_{(p)}: T_p M \rightarrow T_{F(p)}N$ is defined in case

a) by $dF_{(p)}[\sigma] = [F \circ \sigma]$, and in case

b) by $(dF_{(p)} v)[f] = v[F \circ f]$.

2. Let S^2 be the unit sphere in \mathbf{R}^3 , let $j : S^2 \hookrightarrow \mathbf{R}^3$ be the inclusion map, and let

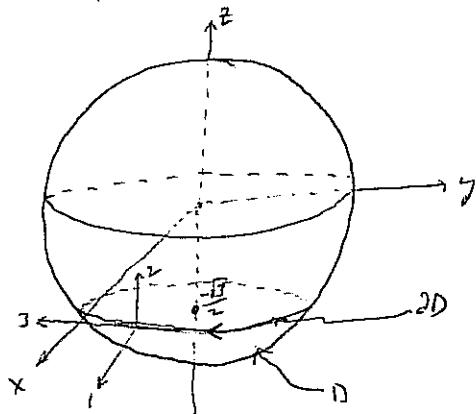
$$D = \{p \in S^2 : z(p) \leq -\frac{\sqrt{3}}{2}\}$$

be an antarctic ice cap. Let S^2 have the orientation defined by an outward pointing normal vector field on S^2 and the standard orientation of \mathbf{R}^3 .

- (a) State Stokes's Theorem for the 1-form $\alpha = j^*(xdy - ydz)$ on D . Explain the correct orientation on the boundary of D .

Stokes's Theorem: $\int_D d\alpha = \int_{\partial D} \alpha$

The orientation of ∂D is as shown here:



- (b) Find $\int_D d\alpha = \int_{\partial D} \alpha$, is easier to calculate.

Parametrize ∂D by $\gamma(\theta) = \left(\frac{\cos \theta}{2}, -\frac{\sin \theta}{2}, -\frac{\sqrt{3}}{2} \right)$, $0 \leq \theta \leq 2\pi$.

$$\gamma^* \alpha = (j \circ \gamma)^* (x dy - y dz) = \frac{\cos \theta}{2} \left(-\frac{\cos \theta}{2} \right) d\theta - 0 = -\frac{\cos^2 \theta}{4} d\theta$$

$$\int_{\partial D} \alpha = \int_0^{2\pi} \gamma^* \alpha = -\frac{1}{4} \int_0^{2\pi} \cos^2 \theta d\theta = -\frac{\pi}{4}.$$

3. Let $j : SL(2; \mathbb{R}) \hookrightarrow \mathbb{R}^{2 \times 2}$ be the inclusion map, and let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be the standard coordinate functions on $\mathbb{R}^{2 \times 2}$. Define differential 2-forms α_j^i , for $1 \leq i, j \leq 2$, on $SL(2; \mathbb{R})$ by

$$\begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} = A^{-1} dA \quad (\text{technically, should be } j^*(A^{-1} dA)).$$

- (a) Find the α_j^i in terms of $j^*dx, j^*dy, j^*dz, j^*dw$. Hint: $A^{-1} = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix}$ on $SL(2; \mathbb{R})$.

$$\begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} = j^* \left(\begin{pmatrix} w & -y \\ -z & x \end{pmatrix} \begin{pmatrix} dx & dy \\ dz & dw \end{pmatrix} \right) = j^* \begin{pmatrix} wdx - ydz & wdy - ydw \\ -zdx + xdz & -zdy + xdw \end{pmatrix}$$

- (b) Prove that $\alpha_1^1 + \alpha_2^2 = 0$ on $SL(2; \mathbb{R})$. Hint: $xw - yz = 1$ on $SL(2; \mathbb{R})$. Understood now that all restricted to $SL(2; \mathbb{R})$, so drop j^* .

$$\alpha_1^1 + \alpha_2^2 = wdx - ydz + (-zdy + xdw) = d(xw - yz) = 0, \text{ since}$$

$xw - yz = 1$ is constant on $SL(2; \mathbb{R})$.

- (c) Knowing that $\alpha_1^1, \alpha_2^1, \alpha_1^2$ is a coframe field on $SL(2; \mathbb{R})$, find $d\alpha_2^1$ in terms of this coframe field. Hint: $d(A^{-1}) = -A^{-1} dA A^{-1}$.

$$\begin{aligned} d \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} &= d(A^{-1} dA) = -A^{-1} dA A^{-1} dA = - \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} \wedge \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} \\ &= - \begin{pmatrix} \alpha_1^1 \wedge \alpha_1^1 + \alpha_2^1 \wedge \alpha_1^2 & \alpha_1^1 \wedge \alpha_2^1 + \alpha_2^1 \wedge \alpha_2^2 \\ \alpha_1^2 \wedge \alpha_1^1 + \alpha_2^2 \wedge \alpha_1^2 & \alpha_1^2 \wedge \alpha_2^1 + \alpha_2^2 \wedge \alpha_2^2 \end{pmatrix} \Rightarrow d\alpha_2^1 = -\alpha_1^1 \wedge \alpha_2^1 - \alpha_2^1 \wedge \alpha_2^2 \\ &\quad = -\alpha_1^1 \wedge \alpha_2^1 + \alpha_2^1 \wedge \alpha_1^1 \text{ by (b)} \\ &\quad = -2\alpha_1^1 \wedge \alpha_2^1 \end{aligned}$$

(d) If the 2-plane distribution \mathcal{D} is defined by

$$\mathcal{D}^\perp = \text{span} \{\alpha_2^1\}$$

verify whether or not \mathcal{D} satisfies the Frobenius condition.

\mathcal{D} satisfies the Frobenius condition iff $\alpha_2^1 \wedge d\alpha_2^1 = 0$.

$$\text{By part (c), } \alpha_2^1 \wedge d\alpha_2^1 = \alpha_2^1 \wedge (-2\alpha_1^1 \wedge \alpha_2^1) = 2\alpha_1^1 \wedge \alpha_2^1 \wedge \alpha_2^1 = 0$$

$$\text{because } \alpha_2^1 \wedge \alpha_2^1 = 0.$$

(e) Given $B \in SL(2; \mathbb{R})$, let $L_B : SL(2; \mathbb{R}) \rightarrow SL(2; \mathbb{R})$ denote left multiplication by B . Prove that $L_B^* \alpha_1^1 = \alpha_1^1$.

$$\begin{aligned} L_B^* \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} &= L_B^* (A^{-1} dA) = (L_B^* A^{-1}) L_B^* dA \\ &= (BA)^{-1} d(BA) = A^{-1} B^{-1} B dA \quad (B \text{ is constant}) \\ &= A^{-1} dA = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 \\ \alpha_1^2 & \alpha_2^2 \end{pmatrix} \Rightarrow L_B^* \alpha_1^1 = \alpha_1^1. \end{aligned}$$

$$\text{Or, directly, } L_B \begin{pmatrix} x & y \\ z & w \end{pmatrix} = B \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} B_1^1 x + B_2^1 z & B_1^1 y + B_2^1 w \\ B_1^2 x + B_2^2 z & B_1^2 y + B_2^2 w \end{pmatrix}$$

$$\begin{aligned} \text{so } L_B^* \alpha_1^1 &= L_B^* (w dx - y dz) = (w \cdot L_B) d(x \cdot L_B) - (y \cdot L_B) d(z \cdot L_B) \\ &= (B_1^2 y + B_2^2 w) d(B_1^1 x + B_2^1 z) - (B_1^1 y + B_2^1 w) d(B_1^2 x + B_2^2 z) \\ &= w dx - y dz, \text{ because } B_1^1 B_2^2 - B_2^1 B_1^2 = 1 \\ &= \alpha_1^1. \end{aligned}$$