Final Examination

Math 441 Tuesday, December 15, 1992

You may quote and use any results that have been proved in class. Make your answers clear and succinct. No credit for answers which I cannot understand in a reasonable amount of time.

1. Let G be a C^∞ manifold which is a group. Suppose that the map $a : G \times G \to G$, $a(x, y) = xy^{-1}$ is C^∞. Prove that the maps $i : G \to G$, $i(x) = x^{-1}$ and $m : G \times G \to G$, $m(x, y) = xy$ are C^∞.

2. Consider the Euclidean group of motions $\mathbf{E}(n)$ as the semidirect product $\mathbf{R}^n \times O(n)$ acting on $\mathbf{R}^n$ by $(a, A)x = a + Ax$. Given $x \in \mathbf{R}^n$, find the isotropy subgroup at x, $\mathbf{E}(n)_x$, and prove that it is isomorphic to $O(n)$.

3. Let $\Gamma = \{\pm \mathrm{id}\}$ act on S^n, where -id is the antipodal map. Prove that Γ acts freely and properly discontinuously.

4. Let $\pi : S^2 \to \mathbf{R}P^2$ be the usual projection map $\pi(x) = [x]$. Consider the vector field on S^2 given by $X(x, y, z) = (-y, x, 0)$. Is X π-related to a vector field Y on $\mathbf{R}P^2$? Hint: how is π-relatedness related to F-invariance, where $F : S^2 \to S^2$ is the antipodal map, $F(p) = -p$?

5. Let X be the vector field on S^2 defined in problem 4. If f is the function on S^2 given by $f(x, y, z) = xy$, find the function Xf.

6. Let $\theta : \mathbf{R} \times S^2 \to S^2$ be the smooth action by $\mathbf{R}$ defined by

$$\theta(t, (x, y, z)) = (x \cos t - y \sin t, x \sin t + y \cos t, z).$$

If X is the infinitesimal generator of θ, what is $X(x, y, z) \in \mathbf{R}^3$ for any $(x, y, z) \in S^2$?

7. Consider the smooth vector field X on S^1 given by $X(x, y) = y(y, -x)$. Let $\theta : W \to S^1$ be its flow.

   (1) Find W.
   (2) Find $\theta(t, (1, 0))$ for any $t \in J_{(1,0)}$.
   (3) Let $U = \{(x, y) \in S^1 : x > 0\}$ and let $y : U \to \mathbf{R}$ be given by $y(x, y) = y$, so that (U, y) is a chart. On U we have $X(x, y) = f(y)\frac{\partial}{\partial y}$. Find $f(y)$.

Do one problem from each of the pairs {8, 8'}, {9, 9'} and {10, 10'}.

8. Let e_0, e_1, e_2, e_3 be the basic orthonormal basis of the quaternions $\mathbf{H}$. Regard the 3-sphere S^3 as the set of quaternions of unit length. Prove the following:

(1) For any $a \in S^3$ the linear transformation $T_a : \mathbf{H} \to \mathbf{H}$ given by $T_a p = ap$ is orthogonal.
(2) For $i = 1, 2, 3$ let $v_i(a) = ae_i$, for any $a \in S^3$. Then v_i is a vector field on S^3.

8'. Exhibit a triangulation of $\mathbf{R}P^2$ and calculate its Euler characteristic.

9. Consider the map

$$F : \mathbf{R}P^1 \times \mathbf{R}P^1 \to \mathbf{R}P^3.$$
$$([x, y], [z, w]) \mapsto [xz, xw, yz, yw]$$

(1) Prove that F is of class C^∞ on some neighborhood of the point $([1, 0], [0, 1]) \in \mathbf{R}P^1 \times \mathbf{R}P^1$.
(2) Calculate the rank of F at this point.

9'. On $\mathbf{R}P^2$ the functions $f[x, y, z] = \frac{y}{x}$ and $g[x, y, z] = \frac{y^2+xz}{x^2}$ are well defined and C^∞ on a neighborhood of the point $[1, 0, 0]$. Show whether f and g are independent at this point (that is, whether $df \wedge dg$ is nonzero at this point).

10. i) State Sard's Theorem.
ii) Let $f : M^m \to N^n$ be a C^1 map and suppose that $m < n$. Using Sard's Theorem, prove that f cannot be surjective.

10'. Let M^2 be a smooth submanifold of $\mathbf{R}^4$. Let $0 \neq v \in \mathbf{R}^4$. Let $v_T : M \to \mathbf{R}^4$ be defined by: for each $p \in M$, $v = v_T(p) + v_\perp(p)$, where $v_T(p) \in T_pM$ and $v_\perp(p) \in T_pM^\perp$. Prove that v_T is a C^∞ vector field on M.