This exam has two parts. The first part is the final exam for Math 442. The second part is the qualifying exam supplement. The Math 441-442 Qualifying Exam comprises both parts. Each problem or problem part is worth 5 points.

Final Exam, Math 442, Spring 2004

1. Let M^n be a connected smooth manifold.

   (a) What is a Riemannian metric on M?

   (b) How is the length of a piecewise smooth curve defined on a Riemannian manifold?

   (c) How is the distance between points defined on a Riemannian manifold?

   (d) For the Riemannian metric

   $$g = \frac{4}{(1+|\mathbf{x}|^2)^2} \sum_1^n dx^i dx^i$$

   on $\mathbf{R}^n$, find the length of the curve

   $$\sigma : [0,1] \to \mathbf{R}^n, \quad \sigma(t) = (t, 0, \ldots, 0)$$

2. Let M^n be a smooth manifold with a linear connection ∇.

   (a) How does ∇ induce the covariant derivative $\frac{\nabla Y}{dt}$ of a vector field Y along a smooth curve σ in M?

   (b) What is the torsion of a linear connection?

   (c) What is the curvature form of a linear connection?

   (d) Define parallel translation along a smooth curve in M.

   (e) On a Riemannian manifold M, g, what linear connection is used and what characterizes it?

3. Consider the unit sphere $S^n \subset \mathbf{R}^{n+1}$, for $n > 1$, with its canonical Riemannian metric g induced from the dot product of $\mathbf{R}^{n+1}$. Let $p = (0, \ldots, 0, 1) \in S^n$ and let $v = \epsilon_1 + \epsilon_2 \in T_pS^n$, where $\epsilon_1, \ldots, \epsilon_{n+1}$ is the standard basis of $\mathbf{R}^{n+1}$. Find $\exp_p v = (a_1, \ldots, a_{n+1})$, that is, find the numbers $a_1, \ldots, a_{n+1}$.

4. Let $H^n = \{(x^1, \ldots, x^n) \in \mathbf{R}^n : x^n > 0\}$ be the upper half space with the Riemannian metric

$$g = \frac{1}{(x^n)^2} \sum_1^n dx^i dx^i$$

   (a) Explain why $\theta^i = \frac{1}{x^n} dx^i$, $i = 1, \ldots, n$ is an orthonormal coframe field on H and find its dual orthonormal moving frame e_i, $i = 1, \ldots, n$, expressed in terms of the standard basis $\epsilon_1, \ldots, \epsilon_n$ of $\mathbf{R}^n$.

   (b) Find the connection forms of the Levi-Civita connection, with respect to this moving frame.

   (c) Let $p = (0, \ldots, 0, 1) \in H$ and let $\gamma : [0, \infty) \to H^n$ be a curve $\gamma(t) = (0, \ldots, 0, f(t))$, for some smooth function on $[0, \infty)$. Find the function $f(t)$ for which $\gamma(t)$ is the geodesic satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = \epsilon_n$.

   (d) Find the Jacobi field $Y(t)$ along the geodesic γ of part (c) for which $Y(0) = 0$ and $Y'(0) = \epsilon_1$.

5. Let $\gamma : [0, a] \to M^n$ be a unit speed geodesic in a Riemannian manifold M, g. If V is a smooth vector field along γ, then the index form is

$$I(V, V) = g(V', V)|_0^a - \int_0^a g(V'' + R(V, \dot{\gamma})\dot{\gamma}, V)\, dt$$

   From this prove that if $V(t)$ is perpendicular to $\dot{\gamma}(t)$ for every t, then

$$I(V, V) = \int_0^a (|V'|^2 - K(V \wedge \dot{\gamma})|V^2|)\, dt$$

   where $K(V \wedge \dot{\gamma})$ is the sectional curvature of the section spanned by $V(t)$ and $\dot{\gamma}(t)$ (and is zero whenever $V(t) = 0$).

6. Let M^n, g be a complete Riemannian manifold. Let $p \in M$ and let $\exp_p : T_pM \to M$ be the exponential map at the point p. Prove that if the sectional curvatures are all less than or equal to zero, then $\exp_p$ has no singularties.

7. Suppose that g is a left-invariant Riemannian metric on the Lie group $SL(n, \mathbf{R})$ for which the Ricci curvature satisfies $S = \lambda g$, for some constant λ; that is, g is an Einstein metric. Explain why it must be the case that $\lambda \leq 0$.

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8. Prove that a flat Riemannian manifold M^n, g is locally isometric to Euclidean space. Hint: for $p \in M$, prove that there exists a local coordinate system $x^1, \ldots, x^n$ on an open set $U \subset M$ about p such that $g = \sum_1^n dx^i dx^i$ on U.

9. Let $\alpha = dx \wedge dy \wedge dz$ be the volume form on Euclidean space R^3. Let $S^2(r)$ be the sphere of radius $r > 0$ centered at the origin. Let N be the unit normal vector field on S^2 whose value at $p \in S^2$ is $N(p) = \frac{1}{r}p$. Then the area form of S^2 for its induced Riemannian metric is given by the interior product

$$\omega = \iota_N dx \wedge dy \wedge dz$$

pulled back to S^2. Explain how Stokes's Theorem relates the volume $V(B(r)) = \int_{B(r)} \alpha$ of the ball $B(r)$ of radius r in $\mathbf{R}^3$ to the area $A(S^2(r)) = \int_{S^2(r)} \omega$ of the sphere $S^2(r)$.

10. Explain how the Gauss-Bonnet Theorem implies that for any Riemannian metric on a smooth surface M homeomorphic to the torus T^2, the Gaussian curvature must be zero at some point of M.