This exam has two parts. The first part is the final exam for Math 442. The second part is the qualifying exam supplement. The Math 441-442 Qualifying Exam comprises both parts. Each problem or problem part is worth 5 points.

Final Exam, Math 442, Spring 2004

1. Let $M^n$ be a connected smooth manifold.

   (a) What is a Riemannian metric on $M$?

   (b) How is the length of a piecewise smooth curve defined on a Rie-
        mannian manifold?

   (c) How is the distance between points defined on a Riemannian man-
        ifold?

   (d) For the Riemannian metric

   $g = \frac{4}{(1 + |x|^2)^2} \sum_{1}^{n} dx^i dx^i$

   on $\mathbb{R}^n$, find the length of the curve

   $\sigma : [0, 1] \rightarrow \mathbb{R}^n, \sigma(t) = (t, 0, \ldots, 0)$

2. Let $M^n$ be a smooth manifold with a linear connection $\nabla$.

   (a) How does $\nabla$ induce the covariant derivative $\frac{\nabla Y}{\partial t}$ of a vector field
        $Y$ along a smooth curve $\sigma$ in $M$?

   (b) What is the torsion of a linear connection?

   (c) What is the curvature form of a linear connection?

   (d) Define parallel translation along a smooth curve in $M$.

   (e) On a Riemannian manifold $M, g$, what linear connection is used
        and what characterizes it?

3. Consider the unit sphere $S^n \subset \mathbb{R}^{n+1}$, for $n > 1$, with its canonical
   Riemannian metric $g$ induced from the dot product of $\mathbb{R}^{n+1}$. Let $p =
   (0, \ldots, 0, 1) \in S^n$ and let $v = \epsilon_1 + \epsilon_2 \in T_p S^n$, where
   $\epsilon_1, \ldots, \epsilon_{n+1}$ is the standard basis of $\mathbb{R}^{n+1}$. Find $\exp_p v =
   (a_1, \ldots, a_{n+1})$, that is, find the numbers $a_1, \ldots, a_{n+1}$.
4. Let \( H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n > 0\} \) be the upper half space with the Riemannian metric
\[
g = \frac{1}{(x^n)^2} \sum_{i=1}^{n} dx^i dx^i
\]
(a) Explain why \( \theta^i = \frac{1}{x^n} dx^i, i = 1, \ldots, n \) is an orthonormal coframe field on \( H \) and find its dual orthonormal moving frame \( e_i, i = 1, \ldots, n \), expressed in terms of the standard basis \( \epsilon_1, \ldots, \epsilon_n \) of \( \mathbb{R}^n \).
(b) Find the connection forms of the Levi-Civita connection, with respect to this moving frame.
(c) Let \( p = (0, \ldots, 0, 1) \in H \) and let \( \gamma : [0, \infty) \to H^n \) be a curve \( \gamma(t) = (0, \ldots, 0, f(t)) \), for some smooth function on \([0, \infty)\). Find the function \( f(t) \) for which \( \gamma(t) \) is the geodesic satisfying \( \gamma(0) = p \) and \( \dot{\gamma}(0) = \epsilon_n \).
(d) Find the Jacobi field \( Y(t) \) along the geodesic \( \gamma \) of part (c) for which \( Y(0) = 0 \) and \( Y'(0) = \epsilon_1 \).

5. Let \( \gamma : [0, a] \to M^n \) be a unit speed geodesic in a Riemannian manifold \( M, g \). If \( V \) is a smooth vector field along \( \gamma \), then the index form is
\[
I(V, V) = g(V', V)|^a_0 - \int_0^a g(V'' + R(V; \dot{\gamma}) \dot{\gamma}, V) dt
\]
From this prove that if \( V(t) \) is perpendicular to \( \dot{\gamma}(t) \) for every \( t \), then
\[
I(V, V) = \int_0^a (|V'|^2 - K(V \wedge \dot{\gamma})|V|^2) dt
\]
where \( K(V \wedge \dot{\gamma}) \) is the sectional curvature of the section spanned by \( V(t) \) and \( \dot{\gamma}(t) \) (and is zero whenever \( V(t) = 0 \)).

6. Let \( M^n, g \) be a complete Riemannian manifold. Let \( p \in M \) and let \( \exp_p : T_p M \to M \) be the exponential map at the point \( p \). Prove that if the sectional curvatures are all less than or equal to zero, then \( \exp_p \) has no singularities.

7. Suppose that \( g \) is a left-invariant Riemannian metric on the Lie group \( SL(n, \mathbb{R}) \) for which the Ricci curvature satisfies \( S = \lambda g \), for some constant \( \lambda \); that is, \( g \) is an Einstein metric. Explain why it must be the case that \( \lambda \leq 0 \).
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8. Prove that a flat Riemannian manifold $M^n$, $g$ is locally isometric to Euclidean space. Hint: for $p \in M$, prove that there exists a local coordinate system $x^1, \ldots, x^n$ on an open set $U \subset M$ about $p$ such that $g = \sum_1^n dx^i dx^i$ on $U$.

9. Let $\alpha = dx \wedge dy \wedge dz$ be the volume form on Euclidean space $\mathbb{R}^3$. Let $S^2(r)$ be the sphere of radius $r > 0$ centered at the origin. Let $N$ be the unit normal vector field on $S^2$ whose value at $p \in S^2$ is $N(p) = \frac{1}{r}p$. Then the area form of $S^2$ for its induced Riemannian metric is given by the interior product

$$\omega = \iota_N dx \wedge dy \wedge dz$$

pulled back to $S^2$. Explain how Stokes’s Theorem relates the volume $V(B(r)) = \int_{B(r)} \alpha$ of the ball $B(r)$ of radius $r$ in $\mathbb{R}^3$ to the area $A(S^2(r)) = \int_{S^2(r)} \omega$ of the sphere $S^2(r)$.

10. Explain how the Gauss-Bonnet Theorem implies that for any Riemannian metric on a smooth surface $M$ homeomorphic to the torus $T^2$, the Gaussian curvature must be zero at some point of $M$. 