Math 442 Final Examination
Geometry Qualifying Examination
May 5, 1993

Instructions The Geometry Qualifying Examination comprises two parts. You must work 8 problems in Part I and 4 problems in Part II. Budget your time accordingly. Part I constitutes the Math 442 Final Examination. Each problem is worth 10 points.

Part I Do all 8 problems in this part.

1. Let G, H be Lie groups with Lie algebras $\mathcal{G}, \mathcal{H}$. Let $f, g : G \to H$ be Lie group homomorphisms, and suppose that G is connected. Prove that if $f_* = g_* : \mathcal{G} \to \mathcal{H}$, then $f = g$.

2. On $\mathbf{R}^3$ let $\mathcal{D}$ be the 2-plane distribution defined by $\mathcal{D}^\perp = \{\theta\}$, where $\theta = dx + z(dy - dz)$. Determine whether or not $\mathcal{D}$ is completely integrable.

3. Let $G = SL(n+1; \mathbf{R})$ and let

$$H = \{\begin{pmatrix} t & {}^t x \\ 0 & B \end{pmatrix} \in G : t \in \mathbf{R}, B \in GL(n; \mathbf{R}), x \in \mathbf{R}^n\}.$$

Prove that G/H is diffeomorphic to the real projective space $\mathbf{R}P^n$.

4. Prove that the DeRham cohomology space $H^1(S^2) = 0$.

5. Fix a point p_0 in the smooth Riemannian surface M, g. Let θ^1, θ^2 be a smooth orthonormal coframe field on M, with respect to which the Levi-Civita connection form is $\omega = \begin{pmatrix} 0 & \omega_2^1 \\ \omega_1^2 & 0 \end{pmatrix}$. Let the Gaussian curvature be K; that is, the curvature form is given by $\Omega_2^1 = K\theta^1 \wedge \theta^2$. Suppose that there exist smooth 1-forms $\omega_1^3 = -\omega_3^1$ and $\omega_2^3 = -\omega_3^2$ on M satisfying (we use the summation convention and the index range $1 \leq i, j, k \leq 2$)

$$\omega_i^3 \wedge \theta^i = 0$$
$$d\omega_i^3 = -\omega_j^3 \wedge \omega_i^j$$
$$\omega_1^3 \wedge \omega_2^3 = K\theta^1 \wedge \theta^2$$

Prove that there exists a neighborhood U of p_0 in M on which there exists a smooth map $F : U \to \mathbf{R}^3$ such that $\langle dF, dF \rangle = g$.

6. Let T^2 be the torus in $\mathbf{R}^3$ obtained by revolving the circle $(x-2)^2 + z^2 = 1$ about the z-axis. If θ is the usual polar angle in the xy-plane and φ is the

angle from the z-axis, then $T^2 = \{2(\cos\theta, \sin\theta, 0) + \cos\varphi(\cos\theta, \sin\theta, 0) + \sin\varphi(0,0,1) : 0 \le \theta \le 2\pi, 0 \le \varphi \le \pi\}$. Let D be the regular domain of integration $D = T^2 \cap \{y \ge 0\}$. Let T^2 be oriented by its outward pointing smooth unit normal vector field e_3.

i) On a sketch, indicate the induced orientation on ∂D.

ii) Use Stokes' Theorem to calculate $\int_D dx \wedge dz$.

7. Consider the upper-half plane $H^2 = \{(x,y) \in \mathbf{R}^2 : y > 0\}$, with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$. Let $\theta^1 = \frac{1}{y}dx$, $\theta^2 = \frac{1}{y}dy$, an orthonormal coframe field on H^2.

i) Find the Levi-Civita connection forms ω^i_j with respect to this coframe field.

ii) Find the curvature forms Ω^i_j.

8. Let ∇ denote the linear connection on the unit sphere S^2 induced from the canonical connection on $\mathbf{R}^3$ by orthogonal projection. Consider the smooth curve $\gamma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, 1)$ in S^2. Verify whether or not γ is a geodesic with respect to this connection.

Part II The following eight problems are grouped in pairs numbered $1, 1', \ldots, 4, 4'$. Do exactly one problem from each such pair, for a total of four problems.

1. Let M be a smooth manifold with smooth vector fields X and Y. Let $\theta : \mathbf{R} \times M \to M$ be the flow of X. Prove that if $[X,Y] = 0$ on M, then Y is θ_t-invariant for every $t \in \mathbf{R}$.

1′. Consider the smooth vector field X on S^2 given by $X(x,y,z) = (y, -x, 0)$. Let $\theta : W \subset \mathbf{R} \times S^2 \to S^2$ be its flow.

i) Find W.

ii) Find $\theta(t, (0,0,1))$ for any $t \in \mathbf{R}$ at which it is defined.

iii) Let $U = S^2 \cap \{z > 0\}$ and let $u : U \to \mathbf{R}^2$ be given by $u(x,y,z) = (x,y)$, so that (U,u) is a chart on S^2. On U we have $X = f\frac{\partial}{\partial u^1} + g\frac{\partial}{\partial u^2}$, for some smooth functions f, g on U. Find f.

2. Let $\Gamma = \{\pm \mathrm{id}\}$ act on S^n, where $-\mathrm{id}$ is the antipodal map. Prove the Γ acts freely and properly discontinuously.

2′. Let $\pi : S^2 \to \mathbf{R}P^2$ be the projection map $\pi(x) = [x]$.

i) Prove that if a vector field X on S^2 is invariant under the antipodal map $A : S^2 \to S^2$, where $Ax = -x$, then X is π-related to a vector field Y on $\mathbf{R}P^2$.

ii) Is the vector field $X_{(x^1,x^2,x^3)} = (-x^2, x^1, x^3)$ on S^2 π-related to a vector field on $\mathbf{R}P^2$? Justify your answer.

3. i) State Sard's Theorem.

ii) Let TM be the tangent bundle of the smooth m-dimensional submanifold M of $\mathbf{R}^n$. Let $F : \mathbf{R} \times TM \to \mathbf{R}^n$ be the map $F(t,p,v) = p + tv$. Prove that if $n > 2m+1$, then there exists a point $q \in R^n$ such that, for each $p \in M$, the line qp from q to p is not tangent to M at p.

3′. Prove that the map $F : \mathbf{R}P^1 \times \mathbf{R}P^1 \to \mathbf{R}P^3$ given by $F([x,y],[z,w]) = [xz, xw, yz, yw]$ is a smooth immersion.

4. Let M be a compact oriented smooth surface embedded in $\mathbf{R}^3$. Let e_3 be a smooth unit normal vector field on M, so that $e_3 : M \to S^2$ is the Gauss map. Prove that $\deg e_3 = \frac{1}{2}\chi(M)$, where $\chi(M)$ is the Euler characteristic of M.

4′. Exhibit a triangulation of the Klein bottle and calculate its Euler characteristic.