Instructions The Geometry Qualifying Examination comprises two parts. You must work 8 problems in Part I and 4 problems in Part II. Budget your time accordingly. Part I constitutes the Math 442 Final Examination. Each problem is worth 10 points.

Part I Do all 8 problems in this part.

1. Let $G, H$ be Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $f, g : G \to H$ be Lie group homomorphisms, and suppose that $G$ is connected. Prove that if $f_* = g_* : \mathfrak{g} \to \mathfrak{h}$, then $f = g$.

2. On $\mathbb{R}^3$ let $\mathcal{D}$ be the 2-plane distribution defined by $\mathcal{D}^\perp = \{\theta\}$, where $\theta = dx + z(dy - dz)$. Determine whether or not $\mathcal{D}$ is completely integrable.

3. Let $G = SL(n + 1; \mathbb{R})$ and let

$$H = \left\{ \begin{pmatrix} t & x \\ 0 & B \end{pmatrix} \in G : t \in \mathbb{R}, B \in GL(n; \mathbb{R}), x \in \mathbb{R}^n \right\}.$$ 

Prove that $G/H$ is diffeomorphic to the real projective space $\mathbb{R}P^n$.

4. Prove that the DeRham cohomology space $H^1(S^2) = 0$.

5. Fix a point $p_0$ in the smooth Riemannian surface $M, g$. Let $\theta^1, \theta^2$ be a smooth orthonormal coframe field on $M$, with respect to which the Levi-Civita connection form is $\omega = \begin{pmatrix} 0 & \omega_2^1 \\ \omega_1^2 & 0 \end{pmatrix}$. Let the Gaussian curvature be $K$; that is, the curvature form is given by $\Omega^2_2 = K\theta^1 \wedge \theta^2$. Suppose that there exist smooth 1-forms $\omega_1^3 = -\omega_3^1$ and $\omega_2^3 = -\omega_3^2$ on $M$ satisfying (we use the summation convention and the index range $1 \leq i,j,k \leq 2$)

$$\omega_i^3 \wedge \theta^i = 0$$

$$d\omega_i^3 = -\omega_j^3 \wedge \omega_i^j$$

$$\omega_1^3 \wedge \omega_2^3 = K\theta^1 \wedge \theta^2$$

Prove that there exists a neighborhood $U$ of $p_0$ in $M$ on which there exists a smooth map $F : U \to \mathbb{R}^3$ such that $\langle dF, dF \rangle = g$.

6. Let $T^2$ be the torus in $\mathbb{R}^3$ obtained by revolving the circle $(x - 2)^2 + z^2 = 1$ about the $z$-axis. If $\theta$ is the usual polar angle in the $xy$-plane and $\varphi$ is the
angle from the z-axis, then $T^2 = \{2(\cos \theta, \sin \theta, 0) + \cos \varphi(\cos \theta, \sin \theta, 0) + 
\sin \varphi(0,0,1) : 0 \leq \theta \leq 2\pi, 0 \leq \varphi \leq \pi\}$. Let $D$ be the regular domain of integration $D = T^2 \cap \{y \geq 0\}$. Let $T^2$ be oriented by its outward pointing smooth unit normal vector field $e_3$.

i) On a sketch, indicate the induced orientation on $\partial D$.

ii) Use Stokes’ Theorem to calculate $\int_D dx \wedge dz$.

7. Consider the upper-half plane $H^2 = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$. Let $\theta^1 = \frac{1}{y}dx$, $\theta^2 = \frac{1}{y}dy$, an orthonormal coframe field on $H^2$.

i) Find the Levi-Civita connection forms $\omega^i_j$ with respect to this coframe field.

ii) Find the curvature forms $\Omega^i_j$.

8. Let $\nabla$ denote the linear connection on the unit sphere $S^2$ induced from the canonical connection on $\mathbb{R}^3$ by orthogonal projection. Consider the smooth curve $\gamma(t) = \frac{1}{\sqrt{2}}(\cos t, \sin t, 1)$ in $S^2$. Verify whether or not $\gamma$ is a geodesic with respect to this connection.

**Part II** The following eight problems are grouped in pairs numbered 1,1', . . . , 4,4'. Do exactly one problem from each such pair, for a total of four problems.

1. Let $M$ be a smooth manifold with smooth vector fields $X$ and $Y$. Let $\theta : \mathbb{R} \times M \to M$ be the flow of $X$. Prove that if $[X,Y] = 0$ on $M$, then $Y$ is $\theta_t$-invariant for every $t \in \mathbb{R}$.

1'. Consider the smooth vector field $X$ on $S^2$ given by $X(x,y,z) = (y,-x,0)$. Let $\theta : W \subset \mathbb{R} \times S^2 \to S^2$ be its flow.

i) Find $W$.

ii) Find $\theta(t,(0,0,1))$ for any $t \in \mathbb{R}$ at which it is defined.

iii) Let $U = S^2 \cap \{z > 0\}$ and let $u : U \to \mathbb{R}^2$ be given by $u(x,y,z) = (x,y)$, so that $(U,u)$ is a chart on $S^2$. On $U$ we have $X = f \frac{\partial}{\partial u^1} + g \frac{\partial}{\partial u^2}$, for some smooth functions $f,g$ on $U$. Find $f$.

2. Let $\Gamma = \{\pm \text{id}\}$ act on $S^n$, where $-\text{id}$ is the antipodal map. Prove the $\Gamma$ acts freely and properly discontinuously.

2'. Let $\pi : S^2 \to \mathbb{R}P^2$ be the projection map $\pi(x) = [x]$.

i) Prove that if a vector field $X$ on $S^2$ is invariant under the antipodal map $A : S^2 \to S^2$, where $Ax = -x$, then $X$ is $\pi$-related to a vector field $Y$ on $\mathbb{R}P^2$. 

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2. Let $\Gamma = \{\pm \text{id}\}$ act on $S^n$, where $-\text{id}$ is the antipodal map. Prove the $\Gamma$ acts freely and properly discontinuously.
ii) Is the vector field \( X_{(x^1, x^2, x^3)} = (-x^2, x^1, x^3) \) on \( S^2 \) \( \pi \)-related to a vector field on \( \mathbb{RP}^2 \)? Justify your answer.

3. i) State Sard’s Theorem.
   ii) Let \( TM \) be the tangent bundle of the smooth \( m \)-dimensional submanifold \( M \) of \( \mathbb{R}^n \). Let \( F : \mathbb{R} \times TM \to \mathbb{R}^n \) be the map \( F(t, p, v) = p + tv \). Prove that if \( n > 2m + 1 \), then there exists a point \( q \in \mathbb{R}^n \) such that, for each \( p \in M \), the line \( qp \) from \( q \) to \( p \) is not tangent to \( M \) at \( p \).

3'. Prove that the map \( F : \mathbb{RP}^1 \times \mathbb{RP}^1 \to \mathbb{RP}^3 \) given by \( F([x, y], [z, w]) = [xz, xw, yz, yw] \) is a smooth immersion.

4. Let \( M \) be a compact oriented smooth surface embedded in \( \mathbb{R}^3 \). Let \( e_3 \) be a smooth unit normal vector field on \( M \), so that \( e_3 : M \to S^2 \) is the Gauss map. Prove that \( \deg e_3 = \frac{1}{2} \chi(M) \), where \( \chi(M) \) is the Euler characteristic of \( M \).

4'. Exhibit a triangulation of the Klein bottle and calculate its Euler characteristic.