FASCICOLO SPECIALE SETTEMBRE 1983

RENDICONTI
DEL
SEMINARIO MATEMATICO

Convegno su

«DIFFERENTIAL GEOMETRY ON HOMOGENEOUS SPACES»

TORINO, 29 SETTEMBRE - 1 OTTOBRE 1983

UNIVERSITA' E POLITECNICO
TORINO

Gary R. Jensen and Marco Rigoli

A CLASS OF HARMONIC MAPS FROM SURFACES INTO REAL GRASSMANNIANS

This note refers to a previous paper of the authors [4] presented in Torino during the 1983 meeting on Differential Geometry held in the "Instituto Matematico" of the University; the reader is therefore referred to [4] for terminology and results.

Throughout this paper we shall call a map $f : M \to S^{2n}$ pseudo-holomorphic if it is a linearly full, isotropic of order n, minimal immersion of an oriented surface M. In §1 we define the j^{th} associated maps

$$f_j : M \to G_{2j+3} (R^{2n+1}) ,$$

and the j^{th} Gauss maps

$$\gamma_j : M \to G_2 (R^{2n+1}) ,$$

for $j = 0, 1, ..., n-1$, where $G_m (R^{2n+1})$ is the Grassmannian of oriented m-dimensional subspaces of R^{2n+1}, $1 \leqslant m \leqslant 2n+1$. We then prove the following two theorems, (compare Barbosa [1], Obata [5] and Ruh-Vilms [6]).

Theorem 1. Let $f : M \to S^{2n}$ be pseudo-holomorphic. Then its j^{th} associated map f_j, $j = 0, 1, ..., n-1$, is harmonic and it is regular and conformal outside the set of zeros (which is isolated) of the $(j+1)^{th}$ contact invariant of f.

Theorem 2. Let $f : M \to S^{2n}$ be pseudo-holomorphic. Then the Gauss maps of f satisfy:

i) γ_0 is regular and conformal on M ; γ_j is regular and conformal outside the set of common zeros (an isolated set) of the j^{th} and $(j+1)^{th}$ contact invariants of f , for $j = 1, \ldots, n-2$; and γ_{n-1} is regular and conformal outside the set of zeros (an isolated set) of the $(n-1)^{th}$ contact invariant of f .

ii) γ_j is harmonic, for $j = 0, 1, \ldots, n-1$.

iii) γ_{n-1} is anti-holomorphic.

In §2 the Riemannian geometry of the real Grassmannians is recalled, then Theorem 1 is proved in §3 and Theorem 2 is proved in §4 .

§1 . Let $f : M \to S^{2n}$ be a pseudo-holomorphic map. In [4] it was shown that generalized n^{th} order frame fields exist about any point of M . We recall that this means that for any point of M there is a neighborhood U about that point on which there exists a map (a frame field along f) $e : U \to$ $\to 0(2n+1)$ characterized by

$$(1) \qquad \begin{cases} \phi_0^\alpha = 0 \ , \ \alpha \geqslant 3 \\ \phi_1^3 = -k_1 \phi^2 + m_1 \phi^2 \ , \ \phi_2^3 = m_1 \phi^1 + k_1 \phi^2 \\ \phi_1^4 = m_1 \phi^1 + k_1 \phi^2 \ , \ \phi_2^4 = k_1 \phi^1 - m_1 \phi^2 \\ \phi_i^\gamma = 0 \ , \ \gamma \geqslant 5 \ ; \end{cases}$$

$$(2) \qquad \begin{cases} i(k_1 + im_1)(\phi_3^5 + i\phi_4^5) = (k_2 + im_2)\phi \\ i(k_1 + im_1)(\phi_3^6 + i\phi_4^6) = (ik_2 - m_2)\phi \\ \phi_{j+2}^\mu = 0 \ , \ \mu \geqslant 7 \ ; \end{cases}$$

and the higher order analogues of (2) , where

$$e^*\Phi_B^A = \phi_B^A \ , \ 0 \leqslant A , B \leqslant 2n$$

are the pull-backs by e of the Maurer-Cartan forms of $0\,(2n+1)$.

These frame fields are determined up to changes $\widetilde{e} = eK$, where K takes values in

(3) $$\widetilde{G}_n = \left\{ \begin{bmatrix} 0 & & & \\ & A_1 & & \\ & & \ddots & \\ & & & A_n \end{bmatrix} : A_k \in SO(2) , \ k = 1, \ldots, n \right\} \subset O(2n+1)$$

Thus with respect to the generalized n^{th} order frame fields $e = (e_0, e_1, \ldots, \ldots, e_{2n})$ the vectors e_{2j+1} , e_{2j+2} are determined up to a rotation, for $j = 1, \ldots, n-1$. Furthermore

(4) $$N_j = \{e_{2j+1} , e_{2j+2}\} , \text{ (i.e., the span) },$$

defines a smooth rank 2 subbundle of the normal bundle $TM^\perp$, which then decomposes into the orthogonal Whitney sum

(5) $$TM^\perp = N_1 \oplus \ldots \oplus N_{n-1} .$$

Definition 1: The map, for $j = 0, 1, \ldots, n-1$,

$$f_j : M \to G_{2j+3} (R^{2n+1}) ,$$

into the Grassmannian of real oriented $(2j+3)$ dimensional subspaces of R^{2n+1} , defined by

$$f_j (p) = \{f(p) , e_1 (p), \ldots, e_{2j+2} (p)\} , \ p \in M ,$$

will be called the j^{th} *associated map* of the pseudo-holomorphic f .

Obata [5] called f_0 the Gauss map of f .

Definition 2: The map, for $j = 0, 1, \ldots, n-1$,

$$\gamma_j : M \to G_2 (R^{2n+1}) ,$$

defined by

$$\gamma_j (p) = \{e_{2j+1} (p) , e_{2j+2} (p)\} , \ p \in M ,$$

will be called the j^{th} Gauss map of f .

§2. In this section we recall some of the geometry of $G_m(R^{2n+1}) = G_m(2n+1)$, $1 \leqslant m \leqslant 2n$. Let ϵ_A , $A = 0, 1, \ldots, 2n$ denote the standard basis of R^{2n+1} . We choose the point $o = \{\epsilon_0, \epsilon_1, \ldots, \epsilon_{m-1}\}$ to be the origin of $G_m(2n+1)$. (Here $\{\}$ denotes the span with the orientation given by this ordered basis). The standard action of $0(2n+1)$ on R^{2n+1} induces a transitive left action on $G_m(2n+1)$. The isotropy subgroup at the origin is

$$G_0 = S0(m) \times 0(2n+1-m) \quad , \tag{6}$$

and thus

$$G_m(2n+1) = 0(2n+1)/G_0 \quad . \tag{7}$$

We adopt the following indexing conventions:

$0 \leqslant A, B, C \leqslant 2n$, $1 \leqslant i, j, k \leqslant m-1$, $0 \leqslant a, b \leqslant m-1$, $m \leqslant \alpha, \beta \leqslant 2n$.

Then Φ^A_B are the matrix entries of Φ . A basis of the annihilator $\mathfrak{g}^{\perp}_0$ is given by $\{\Phi^{\alpha}_a\}$. Thus the quadratic form

$$g = \sum_{\alpha} (\Phi^{\alpha}_0)^2 + \sum_{i,\alpha} (\Phi^{\alpha}_i)^2 \tag{8}$$

on $0(2n+1)$ is $Ad(G_0)$-invariant. It defines an $0(2n+1)$-invariant quadratic form ds^2 on $G_m(2n+1)$, which is, in fact, the standard (up to homothety) invariant Riemannian metric on $G_m(2n+1)$.

To understand the geometry of ds^2 it is convenient to digress a moment to an abstract setting. Let N, ds^2 be a Riemannian manifold and let

$$F(N) \xrightarrow{\pi} N$$

denote its principal bundle of orthonormal frames. Let $\theta = (\theta^p)$ denote the canonical form on $F(N)$, where we adopt for the moment the index convention $1 \leqslant p, q, r \leqslant \dim N = n$. Then the Levi-Civita connection form

$\omega = (\omega^p)$ of ds^2 on $F(N)$ is characterized by the equations

$$(9) \qquad \begin{cases} d\theta^p = -\omega^p_q \wedge \theta^q \\ \omega^p_q + \omega^q_p = 0 \ , \end{cases}$$

plus the well-known properties involving right translations and the value on vertical vectors.

Returning to the geometry of $G_m(2n+1)$, we choose and fix an orthonormal reference frame at o in $G_m(2n+1)$. It is well known that fixing a reference frame at the origin gives rise to a bundle homomorphism

$$\begin{array}{ccc} 0(2n+1) & \rightarrow & F(G_m(2n+1)) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & G_m(2n+1) & \end{array}$$

In this way each element of $0(2n+1)$ is an orthonormal frame of $G_m(2n+1)$, namely the differential of this element applied to the reference frame.

It is an elementary exercise to verify that the canonical form restricted to $0(2n+1)$ is given by

$$(10) \qquad \theta = (\theta^{a,\alpha}) \ , \text{ where } \theta^{a,\alpha} = \Phi^\alpha_a \ .$$

The Maurer-Cartan structure equations of $0(2n+1)$ are

$$(11) \qquad \begin{cases} d\Phi^A_B = -\Phi^A_C \wedge \Phi^C_B \ , \text{ and} \\ \Phi^A_B = -\Phi^B_A \end{cases}$$

Comparing (11) and (9) and using (10) together with the fact that these forms are left-invariant, we conclude that Levi-Civita connection forms $\omega^{a,\alpha}_{b,\beta}$ restricted to $0(2n+1)$ are given by

$$(12) \qquad \begin{cases} \omega^{0,\alpha}_{0,\beta} = \Phi^\alpha_\beta \\ \omega^{0,\alpha}_{i,\beta} = \delta^\alpha_\beta \Phi^0_i \\ \omega^{i,\alpha}_{k,\beta} = \delta^\alpha_\beta \Phi^i_k + \delta^i_k \Phi^\alpha_\beta \end{cases}$$

In preparation for the next section we recall the definition of harmonic map in the abstract setting. (cf. Chern-Goldberg [3]). Let M, dh^2 be another Riemannian manifold, and let $f : M \rightarrow N$ be a smooth map. Suppose that we have a locally defined smooth map $e : M \rightarrow F(N)$ such that $\pi \circ e = f$, (e is a local orthonormal frame field along f). We write

$$e = (f\,; E_1, \ldots, E_n)$$

where, at each point x in the domain of f, $E_1(x), \ldots, E_n(x)$ is an orthonormal basis of $T_{f(x)}N$. Then

$$df = e^*\theta^p \otimes E_p \tag{13}$$

Let ϕ^σ, $\sigma = 1, \ldots, \dim M = m$ be a local orthonormal coframe field in M, so that

$$dh^2 = \Sigma\,(\phi^\sigma)^2 \quad .$$

Let $\phi^\sigma_\tau = -\phi^\tau_\sigma$, $1 \leqslant \sigma, \tau \leqslant m$, denote the Levi-Civita connection forms of dh^2 with respect to this coframe. Then

$$e^*\theta^p = B^p_\sigma\, \phi^\sigma \quad , \tag{14}$$

for certain function B^p in M, and the covariant differential of df is (using (13) and (14))

$$\begin{aligned} Ddf &= dB^p_\sigma \otimes \phi^\sigma \otimes E_p - B^p_\sigma \otimes \phi^\sigma_\tau \otimes \phi^\tau \otimes E_p + B^p_\sigma\, \phi^\sigma \otimes e^*\,\omega^q_p \otimes E_q \\ &= (DB^p_\sigma) \otimes \phi^\sigma \otimes E_p \quad , \end{aligned}$$

where

$$DB^p_\sigma = dB^p_\sigma - B^p_\tau\, \phi^\tau_\sigma + B^q_\sigma\, e^*\,\omega^p_q \overset{\text{def}}{=} B^p_{\sigma\tau}\, \phi^\tau \tag{15}$$

is a 1-form in M. The tension field τ of f is the vector field along f given by

$$\tau = \text{Trace}\ \ Ddf = \sum_\sigma B^p_{\sigma\sigma}\, E_p$$

By definition f is harmonic if $\tau = 0$.

§3 . Proof of Theorem 1.

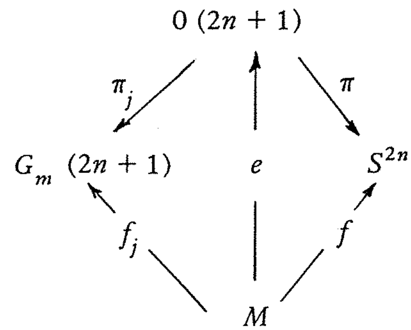
To make our computations we use local n^{th} order frames e along f . These are defined on some neighborhood of any point outside of some isolated set of points in M . They are characterized by

$$
(24) \qquad e^*\Phi = \left|\begin{array}{ccccccccc}
0 & & & & & & & & \\
\varphi^1 & 0 & & & & & & & \\
\varphi^2 & \varphi_1^2 & 0 & & & & & & \\
0 & -r_1\varphi^1 & r_1\varphi^2 & 0 & & & & & \\
\vdots & r_1\varphi^2 & r_1\varphi^1 & \varphi_3^4 & 0 & & & & \\
 & 0 & 0 & r_2\varphi^2 & -r_2\varphi^1 & 0 \ddots & & & \\
 & \vdots & \vdots & r_2\varphi^1 & r_2\varphi^2 & 0 & & & \\
 & & & 0 & 0 & \varphi_{2j-1}^{2j} & 0 & & \\
 & & & \vdots & \vdots & r_j\varphi^2 & -r_j\varphi^1 & 0 & \\
 & & & & & r_j\varphi^1 & r_j\varphi^2 & \varphi_{2j+1}^{2j+2} & 0 \ddots \\
 & & & & & 0 & 0 & & \\
 & & & & & \vdots & \vdots & &
\end{array}\right|
$$

$j = 1, \ldots, n = 1$

(Cf. equation (24) in [4]). The functions r_j are the contact invariants of f. They are globally defined, have isolated zeros (precisely the isolated set referred to above), and $r_j^2 \in C^\infty$ (M) . The forms φ^1 , φ^2 define a local orthonormal coframe on M for the induced metric dh^2 , and φ_2^1 is the Levi-Civita connection form of dh^2 with respect to this coframe.

Now fix $j \in \{0, 1, \ldots, n-1\}$ and put $m = 2j + 3$. The following diagram commutes:



where

$$\pi : (e_0, e_1, \dots\ e_{2n}) \to e_0 \quad ,$$

$$\pi_j : (e_0, e_1, \dots, e_{2n}) \to \{e_0, e_1, \dots, e_{2j+2}\} \ ,$$

and f_j is the j^{th} associated map of f defined in §1 . Hence e is also a local orthonormal frame field along f_j .

Our index conventions are now, for $m = 2j + 3$: $1 \leqslant i, k \leqslant 2j + 2$; $0 \leqslant a, b \leqslant 2j + 2$; $2j + 3 \leqslant \alpha, \beta \leqslant 2n$; $1 \leqslant \mu, \nu \leqslant 2$.

We use (16) and (10) to compute the coefficients in (14) :

$$e^*\theta^{a,\alpha} = B_\mu^{a;\alpha}\ \phi^\mu$$

For $j > 0$:

$$e^*\Phi_0^\alpha = B_\mu^{0,\alpha}\ \phi^\mu = 0$$

$$e^*\Phi_{2j+1}^{2j+3} = B_\mu^{2j+1,2j+3}\ \phi^\mu = r_{j+1}\ \phi^2$$

$$e^*\Phi_{2j+2}^{2j+3} = B_\mu^{2j+2,2j+3}\ \phi^\mu = -r_{j+1}\ \phi^1$$

$$e^*\Phi_{2j+1}^{2j+4} = B_\mu^{2j+1,2j+4}\ \phi^\mu = r_{j+1}\ \phi^1$$

$$e^*\Phi_{2j+2}^{2j+4} = B_\mu^{2j+2,2j+4}\ \phi^\mu = r_{j+1}\ \phi^2$$

$$e^*\Phi_i^\alpha = B_\mu^{i,\alpha}\ \phi^\mu = 0\ , \text{ for } (i,\alpha) \notin S_j\ ,$$

where $S_j = \{(2j+1, 2j+3)\ ,\ (2j+2, 2j+3)\ ,\ (2j+1, 2j+4)\ ,\ (2j+ + 2, 2j+4)\}$.

For $j=0$:

(18)
$$\begin{aligned}
e^*\Phi_0^\alpha &= B_\mu^{0,\alpha}\phi^\mu = 0\\
e^*\Phi_1^3 &= B_\mu^{1,3}\,\phi^\mu = -r_1\phi^1\\
e^*\Phi_2^3 &= B_\mu^{2,3}\phi^\mu = r_1\ \phi^2\\
e^*\Phi_1^4 &= B_\mu^{1,4}\phi^\mu = r_1\phi^2\\
e^*\Phi_2^4 &= B_\mu^{2,4}\phi^\mu = r_1\phi^1\\
e^*\Phi_i^\alpha &= B_\mu^{i,\alpha}\ \phi^\mu = 0\ ,\quad \text{for}\quad (i,\alpha)\notin S_0\ ,
\end{aligned}$$

where $S_0 = \{(1,3), (2,3), (1,4), (2,4)\}$. In (17) and (18) the functions $B_\mu^{a,\alpha}$ are defined by the last equation in each line.

From (17), (18) and (8) we have for any $j \geqslant 0$

(19)
$$f_j^*ds^2 = e^*g = 2r_{j+1}^2\ ((\phi^1)^2 + (\phi^2)^2) = 2r_{j+1}^2\ db^2\ .$$

Hence the j^{th} associated map f_j is regular and conformal outside of the set of zeros of the $(j+1)^{st}$ order contact invariant r_{j+1} of f .

To show now that f_j is harmonic we must compute its tension field τ , and this requires that we compute the covariant derivatives of the coefficients $B^{a,\alpha}$, (cf. §2) .

Using equations (15), (16), (10), (12), (17) and (18) we obtain

(20)
$$\begin{aligned}
DB_\mu^{0,\alpha} = {}& B_\mu^{2j+1,2j+3}\ \delta_{2j+3}^\alpha\ \phi_{2j+1}^0 + B_\mu^{2j+2,2j+3}\ \delta_{2j+3}^\alpha\ \phi_{2j+2}^0\\
&+ B_\mu^{2j+1,2j+4}\,\delta_{2j+4}^\alpha\ \phi_{2j+1}^0 + B_\mu^{2j+2,2j+4}\ \delta_{2j+4}^\alpha\ \phi_{2j+2}^0,
\end{aligned}$$

where we write $\phi_B^A = e^*\Phi_B^A$.

Hence, if $\alpha \neq 2j+3$ or $2j+4$, then

(21)
$$B_{\mu\nu}^{0,\alpha} = 0$$

where these coefficients are defined by the second equation in (15) . If $a = 2j + 3$

$$DB_{\mu}^{0,\alpha} = - B_{\mu}^{2j+1,2j+3}\,\phi_0^{2j+1} - B_{\mu}^{2j+2,2j+3}\,\phi_0^{2j+2} ,$$

and if $\alpha = 2j + 4$

$$DB_{\mu}^{0,\alpha} = - B_{\mu}^{2j+1,2j+4}\,\phi_0^{2j+1} - B_{\mu}^{2j+2,2j+4}\,\phi_0^{2j+2} .$$

In either case, by (16),

(22) $$B_{\mu\nu}^{0,\alpha} = 0 \quad \text{for} \quad j > 0 .$$

If $j = 0$, then

(23) $$B_{\mu\nu}^{0,3} = B_{\mu}^{\nu,3} , \quad B_{\mu\nu}^{0,4} = - B_{\mu}^{\nu,4} .$$

Thus for $0 \leqslant j \leqslant n - 1$, we have from (18), (21), (22), and (23) and the definition of the tension field

(24) $$\tau^{0,\alpha} = \sum_{\mu} B_{\mu\mu}^{0,\alpha} E_{0,\alpha} = 0 .$$

Again using (15), (16), (10), (12), (17) and (18) we have

(25) $$\begin{aligned} DB_{\mu}^{i,\alpha} = dB_{\mu}^{i,\alpha} - B_{\nu}^{i,\alpha}\,\phi_{\mu}^{\nu} &+ B_{\mu}^{2j+1,2j+3}\,(\delta_{2j+3}^{\alpha}\,\phi_{2j+1}^{i} + \delta_{2j+1}^{i}\,\phi_{2j+3}^{\alpha}) \\ &+ B_{\mu}^{2j+2,2j+3}\,(\delta_{2j+3}^{\alpha}\,\phi_{2j+2}^{i} + \delta_{2j+2}^{i}\,\phi_{2j+3}^{\alpha}) \\ &+ B_{\mu}^{2j+1,2j+4}\,(\delta_{2j+4}^{\alpha}\,\phi_{2j+1}^{i} + \delta_{2j+1}^{i}\,\phi_{2j+4}^{\alpha}) \\ &+ B_{\mu}^{2j+2,2j+4}\,(\delta_{2j+4}^{\alpha}\,\phi_{2j+2}^{i} + \delta_{2j+2}^{i}\,\phi_{2j+4}^{\alpha}) . \end{aligned}$$

We begin by considering the case $(i, \alpha) \in S_j$, where S_j was defined in (17) and (18). We recall the following formula from [4] (equation $(30)_j$)

(26) $$*d \log r_{j+1} = \phi_2^1 - \phi_{2j+2}^{2j+1} + \phi_{2j+4}^{2j+3} ,$$

where $*$ is the Hodge star operator on M. We consider the case $(i, \alpha) = (2j+1, 2j+3)$. The remaining cases are completely similar and we will simply state the results.

If $j > 0$, then from (25), (17) and (26) we obtain

(27) $$DB_1^{i,\alpha} = r_{j+1} \, (*d \log r_{j+1}) = *dr_{j+1} \;,$$

and from (25) and (17) we have

(28) $$DB_2^{i,\alpha} = dr_{j+1} \;.$$

Hence, if $dr_{j+1} = r_{j+1,\mu} \, \phi^\mu$, then from (27) and (28)

$$B_{11}^{i,\alpha} = - r_{j+1,2} \;, \quad B_{22}^{i,\alpha} = r_{j+1,2} \;,$$

and thus

(29) $$\tau^{2j+1,2j+3} = \sum_\mu B_{\mu\mu}^{i,\alpha} = 0 \;.$$

In the same way

(30) $$\tau^{i,\alpha} = 0 \;, \quad \text{for} \;\; j = 0, 1, \dots, n-1 \;, \;\; (i,\alpha) \in S_j \;.$$

We are left with the case $(i,\alpha) \notin S_j$. We do here the cases $j > 0$. The results are the same when $j = 0$. Now, from (17),

(31) $$B_\mu^{i,\alpha} = 0 \;.$$

From (31) and (25) we deduce that if $i \neq 2j+1$ or $2j+2$, and $\alpha \neq 2j+3$ or $2j+4$, then

(32) $$DB_\mu^{i,\alpha} = 0 \;, \quad \text{and} \qquad B_{\mu\nu}^{i,\alpha} = 0 \;.$$

The remaining cases are similar to the case

$$i = 2j+1 \;\; \text{and} \;\; \alpha \neq 2j+3 \;\; \text{or} \;\; 2j+4 \;,$$

which we shall show here. From (25) and (31)

$$DB_{\mu}^{2j+1,\alpha} = B_{\mu}^{2j+1,2j+3}\,\phi_{2j+3}^{\alpha} + B_{\mu}^{2j+1,2j+4}\,\phi_{2j+4}^{\alpha} \quad . \tag{33}$$

Thus, from (16), if $\alpha > 2j+6$, then

$$DB_{\mu}^{2j+1,\alpha} = 0$$

and hence

$$B_{\mu\nu}^{2j+1,\alpha} = 0 \quad . \tag{34}$$

The cases $\alpha = 2j+5$ and $\alpha = 2j+6$ remain. From (33), (16) and (17)

$$B_{11}^{2j+1,2j+5} = -r_{j+1}\,r_{j+2} \;,\; B_{22}^{2j+1,2j+5} = r_{j+1}\,r_{j+2} \tag{35}$$

$$B_{11}^{2j+1,2j+6} = 0 = B_{22}^{2j+1,2j+6} \tag{36}$$

Thus from (36), (35), (34) and (32) we deduce

$$\tau^{i,\alpha} = 0 \quad , \quad \text{for} \quad (i,\alpha) \notin S_j \;,\; j \geqslant 0 \;. \tag{37}$$

Equations (24), (30) and (37) complete the proof of Theorem 1.

§4 . Proof of Theorem 2.

We begin by repeating much of §2 for the case of $G_2\,(2n+1)$. It is convenient to reformulate everything for different choices of origin of $G_2\,(2n+1)$.

Fix $j \in \{0, \ldots, n-1\}$. As in §2 we let ϵ_A , $A = 0, 1, \ldots, 2n$ denote the standard ordered basis of R^{2n+1} , but now we choose the point

$$o_j = \{\epsilon_{2j+1} \,,\, \epsilon_{2j+2}\}$$

as the origin of $G_2\,(2n+1)$. Let G_j denote the isotropy subgroup at o_j

of $0(2n+1)$, and let g_j denote its Lie algebra.

We adopt the following indexing convention: $1 \leqslant M, R \leqslant 2j$; $2j + 1 \leqslant s, t \leqslant 2j+2$; $2j+3 \leqslant \alpha, \beta \leqslant 2n$; where if $j = 0$ then M, R are vacuous, and if $j = n-1$ then α, β are vacuous. We continue the convention $0 \leqslant A, B \leqslant 2n$ of §2.

A basis of the annihilator $g_j^{\perp}$ is $\{\Phi_0^s, \Phi_M^s, \Phi_s^\alpha\}$ where $\Phi = (\Phi_B^A)$ is the Maurer-Cartan form of $0(2n+1)$.

The quadratic form

$$g_j = \sum_s (\Phi_0^s)^2 + \sum_{s,M} (\Phi_M^s)^2 + \sum_{a,s} (\Phi_s^\alpha)^2 \tag{38}$$

on $0(2n+1)$ is $Ad(G_j)$-invariant, and thus defines an $0(2n+1)$-invariant quadratic tensor ds^2 on $G_2(2m+1)$, which is the standard invariant Riemannian metric there, up to homothety.

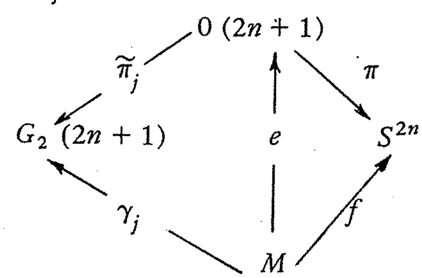
As we saw in (10), we see that the components of the canonical form of $G_2(2n+1)$ restricted to $0(2n+1)$ are given by

$$\theta^{0,s} = \Phi_0^s \,, \quad \theta^{M,s} = \Phi_M^s \,, \quad \theta^{s,\alpha} = \Phi_s^\alpha \,. \tag{39}$$

Using the structure equations (11) we again deduce, as we did in (12), that the Levi-Civita connection forms of g_j on $0(2n+1)$ are given by:

$$\begin{aligned}
\omega_{0,t}^{0,s} &= \Phi_t^s \\
\omega_{M,t}^{0,s} &= \delta_t^s \, \Phi_M^0 \\
\omega_{t,\alpha}^{0,s} &= -\,\delta_t^s \Phi_\alpha^0 \\
\omega_{R,t}^{M,s} &= \delta_R^M \, \Phi_t^s + \delta_t^s \, \Phi_R^M \\
\omega_{t,\alpha}^{M,s} &= -\,\delta_t^s \, \Phi_\alpha^M \\
\omega_{t,\beta}^{s,\alpha} &= \delta_t^s \, \Phi_\beta^\alpha + \delta_\beta^\alpha \, \Phi_t^s
\end{aligned} \tag{40}$$

Fix $j \in \{0, 1, \ldots, n-1\}$ and let $e = (f; e_1, \ldots, e_{2n})$ be a local n^{th} order frame field along f. The following diagram commutes, as can be seen from the definition of γ_j in §1 :



where

$$\pi : A = (e_0, \ldots, e_{2n}) \to e_0$$

$$\widetilde{\pi}_j : A \to \{e_{2j+1}, e_{2j+2}\} \ ,$$

and γ_j is the j^{th} Gauss map of f .

As in §3 we use the convention $1 \leqslant \mu , \nu \leqslant 2$. We use (16) and (39) to compute the coefficients in (14):

$$e^*\theta^{0,s} = B^{0,s}_\mu \, \phi^\mu \quad , \quad e^*\theta^{M,s} = B^{M,s}_\mu \, \phi^\mu \quad , \quad e^*\theta^{s,\alpha} = B^{s,\alpha}_\mu \, \phi^\mu$$

These are:
For $j = 2, \ldots, n-2$:

$$B^{0,s}_\mu = 0$$

$$B^{2j-1,2j+1}_\mu = r_j \, \delta^2_\mu \quad , \quad B^{2j,2j+1}_\mu = -r_j \, \delta^1_\mu$$

$$B^{2j-1,2j+2}_\mu = r_j \, \delta^1_\mu \quad , \quad B^{2j,2j+2}_\mu = r_j \, \delta^2_\mu$$

(41) $$B^{M,s}_\mu = 0 \ , \quad \text{for} \quad (M,s) \notin H_j$$

$$B^{2j+1,2j+3}_\mu = r_{j+1} \, \delta^2_\mu , \quad B^{2j+2,2j+3}_\mu = -r_{j+1} \, \delta^1_\mu$$

$$B^{2j+1,2j+4}_\mu = r_{j+1} \, \delta^1_\mu , \quad B^{2j+2,2j+4}_\mu = r_{j+1} \, \delta^2_\mu$$

$$B^{s,\alpha}_\mu = 0 \ , \quad \text{for} \quad (s, \alpha) \notin S_j \ ,$$

where

$$H_j = \{(2j-1,\, 2j+1)\,,\,(2j\,,2j+1)\,,\,(2j-1,2j+2)\,,\,(2j\,,\,2j+2)\}\,,$$

and S_j is defined in (17).

For $j = 1$:

$$\begin{aligned}
&B^{0,s}_{\mu} = 0 \\
&B^{1,3}_{\mu} = -r_1\,\delta^1_{\mu} \quad , \quad B^{2,3}_{\mu} = r_1\,\delta^2_{\mu} \\
&B^{1,4}_{\mu} = r_1\,\delta^2_{\mu} \quad , \quad B^{2,4}_{\mu} = r_1\,\delta^1_{\mu} \\
(41)_1 \qquad &B^{M,s}_{\mu} = 0 \quad , \quad \text{for} \quad (M\,,\,s) \notin H_1 \\
&B^{3,5}_{\mu} = r_2\,\delta^2_{\mu} \quad , \quad B^{4,5}_{\mu} = -r_2\,\delta^1_{\mu} \\
&B^{3,6}_{\mu} = r_2\,\delta^1_{\mu} \quad , \quad B^{4,6}_{\mu} = r_2\,\delta^2_{\mu} \\
&B^{s,\alpha}_{\mu} = 0 \quad , \quad \text{for} \quad (s\,,\,\alpha) \notin S_1
\end{aligned}$$

where

$$H_1 = \{(1{,}3)\,,\,(2{,}3)\,,\,(1{,}4)\,,\,(2{,}4)\}\,,$$

and S_1 is defined in (17).

For $j = 0$, (so the M index is vacuous):

$$\begin{aligned}
&B^{0,1}_{\mu} = \delta^1_{\mu},\; B^{0,2}_{\mu} = \delta^2_{\mu} \\
(41)_0 \qquad &B^{1,3}_{\mu} = -r_1\,\delta^1_{\mu}\;;\; B^{2,3}_{\mu} = r_1\,\delta^2_{\mu} \\
&B^{1,4}_{\mu} = r_1\,\delta^2_{\mu}\;,\; B^{2,4}_{\mu} = r_1\,\delta^1_{\mu} \\
&B^{s,\alpha}_{\mu} = 0\;,\; \text{if} \quad \alpha \geqslant 5\;.
\end{aligned}$$

For $j = n-1$, when $n = 2$, the coefficients are given by the first three rows of $(41)_1$, and when $n \geqslant 3$:

$$(41)_{n-1} \qquad \begin{aligned} &B_\mu^{0,s} = 0 \\ &B_\mu^{M,s} = 0 \quad \text{if} \quad M \leqslant 2n-4 \ , \\ &B_\mu^{2n-3,2n-1} = r_{n-1} \ \delta_\mu^2 \ , \quad B_\mu^{2n-2,2n-1} = -r_{n-1} \ \delta_\mu^1 \\ &B_\mu^{2n-3,2n-1} = r_{n-1} \ \delta_\mu^1 \ , \quad B_\mu^{2n-2,2n} = r_{n-1} \ \delta_\mu^2 \ , \end{aligned}$$

(which are all the cases since here the α index is vacuous)

From (38) and (41) if follows that

$$\begin{aligned} &\gamma_0^* \ ds^2 = e^* g_0 = (1 + 2r_1^2)\, dh^2 \\ &\gamma_j^* \ ds^2 = e^* g_j = 2(r_j^2 + r_{j+1}^2)\, dh^2 \ , \ j = 1, \dots, n-2 \\ &\gamma_{n-1}^* ds^2 = e^* g_{n-1} = 2r_{n-1}^2 \ dh^2 \ ; \end{aligned}$$

where, as above, $dh^2 = (\phi^1)^2 + (\phi^2)^2$ is the metric on M . This proves (i) of Theorem 2.

To compute the tension field τ of γ_j we proceed exactly as we did in §3 to prove Theorem 1. The result is that $\tau = 0$. We will illustrate the computations, which all use (40), (41), (16) and (26), in just one case. For $2 \leqslant j \leqslant n-2$:

$$\begin{aligned} DB_\mu^{2j-1,2j+2} &= dB_\mu^{2j-1,2j+2} - B_\nu^{2j-1,2j+2}\, \phi_\mu^\nu \\ &\quad + B_\mu^{M,t}\, \omega_{M,t}^{2j-1,2j+2} + B_\mu^{t,\alpha}\, \omega_{t,\alpha}^{2j-1,2j+2} \\ &= dB_\mu^{2j-1,2j+2} - B_\nu^{2j-1,2j+2} \phi_\mu^\nu + B_\mu^{2j-1,2j+1}\, \omega_{2j-1,2j+1}^{2j-1,2j+2} \\ &\quad + B_\mu^{2j,2j+2}\, \omega_{2j,2j+2}^{2j-1,2j+2} + B_\mu^{2j+2,2j+3}\, \omega_{2j+2,2j+3}^{2j-1,2j+2} + B_\mu^{2j+2,2j+3}\, \omega_{2j+2,2j+4}^{2j-1,2j+2} \\ &= dB_\mu^{2j-1,2j+2} - B_\nu^{2j-1,2j+2} \phi_\mu^\nu + B_\mu^{2j-1,2j+1}\, \phi_{2j+1}^{2j+2} - B_\mu^{2j,2j+2}\, \phi_{2j-1}^{2j} \end{aligned}$$

Therefore

$$(43) \qquad DB_1^{2j-1,2j+2} = dr_j \ ,$$

(44) $$DB_2^{2j-1,2j+2} = -(*dr_j) \quad .$$

From (43) and (44) we deduce

(45) $$\tau^{2j-1,2j+2} = 0 \quad .$$

The remaining cases go in the same way. This proves (*ii*) of Theorem 2 .

To prove (*iii*) of Theorem 2 we need to examine the complex structure of $G_2(2n+1)$. This is easily understood in terms of its standard imbedding into CP^{2n} : the oriented plane $\{v_1, v_2\}$ with oriented orthonormal basis v_1, v_2 goes to $[v_1 + iv_2] \in CP^{2n}$. Here we let

$$p : C^{2n+1} \setminus \{0\} \to CP^{2n}$$

denote the standard projection and write

$$p(v) = [v] \quad .$$

Let $e = (e_0, e_1, ..., e_{2n})$ be a local n^{th} order frame field along f . As a mapping into CP^{2n} the j^{th} Gauss map γ_j is given by

$$\gamma_j = [e_{2j+1} + ie_{2j+2}] \quad , \quad j = 0, 1, ..., n-1 \quad .$$

Consider the locally defined map

(46) $$E_j : M \to C^{2n+1}$$
$$E_j = e_{2j+1} + ie_{2j+2} \quad .$$

Then, on the domain of e , $\gamma_j = p \circ E_j$.

Using (16) and the structure equations (11) we obtain what Chern calls the Frenet-Boruvka formulas for f

$$dE_0 = -\phi e_0 + i\phi_2^1 E_0 - r_1 \bar{\phi}\bar{E}_1$$

$$dE_j = -ir_j \bar{\phi} E_{j-1} + i\phi_{2j+2}^{2j+1} E_j - ir_{j+1} \phi E_{j+1} \quad ,$$

(47) for $j = 2, ..., n-1$ if $n \geqslant 3$;

$$dE_1 = r_1\,\overline{\phi}\overline{E}_0 + i\phi_4^3 E_1 - ir_2\,\phi E_2 \quad , \text{ if } \; n \geqslant 3 \; ,$$

$$dE_{n-1} = i\phi_{2n}^{2n-1}\, E_{n-1} - ir_{n-1}\,\overline{\phi}\, E_{n-2} \; , \quad \text{for} \; n \geqslant 2 \; .$$

Then

$$d\gamma_j = dp_{E_j} \circ dE_j \tag{48}$$

where dp_{F_j} denotes the differential of p at the point $E_j \in C^{2n+1}$.

Using the fact that

$$dp_v\, v = 0 \quad , \quad \text{for any} \quad v \in C^{2n+1} \setminus \{0\} \; ,$$

we conclude from (47) and (48) that γ_{n-1} is anti-holomorphic, while the γ_j for $j \leqslant n-2$ are neither holomorphic nor anti-holomorphic. This completes the proof of Theorem 2.

REFERENCES

[1] K. J. Barbosa, *On minimal immersions of* S^2 *into* S^{2m} , Trans AMS 210 (1975), 75-106.

[2] S.S. Chern, *On the minimal immersions of the two sphere in a space of constant curvature*, Problems in Analysis, Princeton, N. J., 1970, 27-40.

[3] S. S. Chern, and S. I. Goldberg, *On the volume decreasing property of a class of real harmonic mappings.* Am. J. of Math. 97 (1975) 133-147.

[4] G. Jensen, and M. Rigoli, *Minimal Surfaces in Spheres,* Rendiconti Sem. Mat. Fis., Torino 1983, 75-98.

[5] M. Obata, *The Gauss map of immersions of Riemannian manifolds in spaces of constant curvature.* J. Diff. Geom. 2 (1968) 217-223.

[6] E. A. Ruh and J. Vilms, *The tension field of the Gauss map,* Trans AMS 149 (1970), 569-573.